Basis and Dimension

DEFINITION

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to form a **basis** for V if
(a) $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = V$ and
(b) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Note: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for a vector space V, then
(a) they must be **nonzero** ← WHY?
and
(b) they must be **distinct**. ← WHY?

EXAMPLES

- $i = \begin{bmatrix}1\\0\end{bmatrix}$ and $j = \begin{bmatrix}0\\1\end{bmatrix}$ are a basis for R^2 ← called the **natural basis** for R^2
- $i = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $j = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $k = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are a basis for R^3 ← called the **natural basis** for R^3
- $e_1 = \text{col}_1(I_n), e_2 = \text{col}_2(I_n), \ldots, e_n = \text{col}_n(I_n)$ are a basis for R^n ← called the **natural basis** for R^n
- $\{t, 1\}$ is called the **natural basis** for P_1
- $\{t^2, t, 1\}$ is called the **natural basis** for P_2
- $\{t^3, t^2, t, 1\}$ is called the **natural basis** for P_3
- $\{t^n, t^{n-1}, \ldots, t, 1\}$ is called the **natural basis** for P_n
- $\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$ is called the **natural basis** for M_{22}

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},$$
$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$
is the **natural basis** for M_{33}

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Example Show that $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for R^3.

We have to show that S is linearly independent and that span(S) = R^3.
So we need to consider the solutions of two linear systems:

System $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$ must have only the zero solution.

System $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 0 & c \end{array}\right]$ must be consistent for any values of a, b, c.

Combine these into a system with two augmented columns and then find the RREF of only the coefficient matrix, not the augmented columns.

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 0 & a \\ 1 & 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 & c \end{array}\right] \quad \text{The RREF is} \quad \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 0 & b-c \\ 0 & 1 & 0 & 0 & c \\ 0 & 0 & 1 & 0 & a-b+c \end{array}\right]$$

We see that both conditions are satisfied.

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How could the preceding example have been done another way?

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Example Show that set $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for the subspace W of R^4 consisting of all vectors of the form $\begin{bmatrix} a+c \\ b-a \\ b-c \\ a+b \end{bmatrix}$ where a, b, and c are any real numbers.

Proceeding as in the previous example we have the following matrix with two augmented columns:

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 0 & a+c \\ -1 & 1 & 0 & 0 & b-a \\ 0 & 1 & -1 & 0 & b-c \\ 1 & 1 & 0 & 0 & a+b \end{array}\right] \quad \text{whose RREF is} \quad \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 0 & b \\ 0 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & a+c-b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

<u>Note the 2nd augmented column must be an arbitrary member of the subspace W.</u>

How do you find a basis for the row space of a matrix?

How do you find a basis for the column space of a matrix?

How do you find a basis for the null space of A?

Some FACTS about BASES

- A vector space *V* is called **finite-dimensional** if there is a finite subset of *V* that is a basis for *V.* If there is no such finite subset of *V,* then *V* is called **infinite-dimensional.**

- If S = {$\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_n$} is a basis for a vector space V, then every vector in V can be written in **one and only one way** as a linear combination of the vectors in S.

- Let S = {$\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_n$} be a set of nonzero vectors in a vector space V and let W = span S. Then some subset of S is a basis for W.

    This says a spanning set for a vector space contains a basis.

- If S = {$\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_n$} is a basis for a vector space V and T = {$\mathbf{w}_1$, $\mathbf{w}_2$, . . . , $\mathbf{w}_r$} is a linearly independent set of vectors in V, then $r \leq n$.

    This says that the largest linearly independent set in a vector space has the number of vectors in a basis.

- If S = {$\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_n$}and T = {$\mathbf{w}_1$, $\mathbf{w}_2$, . . . , $\mathbf{w}_m$} are bases for a vector space, then n = m.

    This says that every basis for a vector space has the same number of vector in it.

DEFINITION
The **dimension** of a nonzero vector space *V* is the number of vectors in a basis for *V.*

We often write **dim(*V*)** for the dimension of *V.*

Since the set **{0}** is linearly dependent, it is natural to say that the vector space **{0}** has dimension **zero.**

dim(R^2) = 2 dim(P_2) = 3 dim(M_{22}) = 4

Examples:

1. Is the set S = {(4,9), (1,2), (-1,3)} a basis for R^2?
2. Is the set S = {(1,2,1), (8,-1,0)} a basis for R^3?
3. Is {t+2,t-1, 4} a basis for P_1?
4. Is the set S = $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for R^4?

DEFINITION

If A is an $m \times n$ matrix, we refer to the dimension of the null space of A as the **nullity** of A, denoted by nullity A.

The nullity of a matrix A is the number of arbitrary constants in the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

- If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V which contains S.

  This says that a linearly independent set of vectors in a vector space *V* can be extended to a basis for *V.*

- Let V be an n-dimensional vector space, and let S = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors in V.

  (a) If S is linearly independent, then it is a basis for V.

  (b) If S spans V, then it is a basis for V.

  This says if you know that the vector space V has dim(V) = k, and you know a set S has exactly k vectors and they are linearly independent, then it is a basis for V.

  This says if you know that the vector space V has dim(V) = k, and you know a set S has exactly k vectors and they span V, then it is a basis for V.

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Let V be a vector space and S a set of vectors from V. Let W = span(S) which is a subspace of V. The set S is a spanning set of W and it must contain a basis for W. There is a technique for determining which members of S form a basis for W that uses the RREF of a homogeneous system.

Summary: Write a linear combination of the members of S with unknown coefficients and set it equal to the zero vector. Form the homogeneous linear system corresponding to the linear combination. Find RREF of the augmented matrix. **The leading 1's in the RREF "point" to the vectors of the set that are linearly independent**, and hence a basis for W.

Example Let $S = \{(1,2,-2,1),(-3,0,-4,3),(7,2,6,-5),(1,0,0,1)\}$. Then W = span(S) is a subspace of R^4. Determine a subset of set S that is a basis for W.

We form the linear combination

$$c_1(1,2,-2,1)+c_2(-3,0,-4,3)+c_3(7,2,6,-5)+c_4(1,0,0,1)=(0,0,0,0).$$

Expanding, adding corresponding entries, and equating corresponding entries we are led to the linear system that can be written in matrix notation shown next.

$$\begin{bmatrix} 1 & -3 & 7 & 1 \\ 2 & 0 & 2 & 0 \\ -2 & -4 & 6 & 0 \\ 1 & 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We form the augmented matrix and find its RREF which is

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading 1's **"point"** to the first vector, the second vector, and the fourth vector. Hence $T = \{(1,2,-2,1),(-3,0,-4,3),(1,0,0,1)\}$ is a basis for W.

Rank of a Matrix

Let **A** be an m by n matrix.

- The row space of **A** is a subspace of R^n and its dimension is called the **row rank of A**.
- The column space of **A** is a subspace of R^m and its dimension is called the **column rank of A**.
- It can be shown that the row rank of **A** = column rank of **A**, so we shorten this and just talk about the **rank of matrix A**.
- An easy way to think about the rank of matrix is as follows:

  rank(A) = the number of nonzero rows in the RREF of A
  = the number of leading 1's in the RREF of A

Facts about rank

Let **A** be an **n by n** matrix.

A is nonsingular if and only if **rank(A) = n**.

Rank(A) = n if and only if **det(A) ≠ 0**.

Linear system **Ax = b** has a unique solution if an only if **rank(A) = n**.

Homogeneous linear system **Ax = 0** has a nontrivial solution if and only if **rank(A) < n**.

List of Nonsingular Equivalences

The following statements are equivalent for an $n \times n$ matrix A.

1. A is nonsingular.
2. $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\det(A) \neq 0$.
6. A has rank n.
7. A has nullity 0.
8. The rows of A form a linearly independent set of n vectors in R^n.
9. The columns of A form a linearly independent set of n vectors in R^n.

Assignment for Exercises on Basis, Dimension and Rank

Page 314 Section 6.4 in the text

#2a, b, c, 6, 8b, 17a, 18a, c, 19b, 20a, 31, 32, 33

Page 338, Section 6. 6 #5a, 6a, 13, 14 (for 13 & 14, just compute rank(A) and determine the dimension of the null space of A)