CONNECTIONS

Matrices-Vector Spaces-Span-LI/LD-Basis-Dimension-Rank

- A set of vectors S = {v_1, v_2, …, v_k} **spans** a vector space V if and only if every vector w in V is a linear combination of the members of S; $w = \sum_{i=1}^{k} c_i v_i$ where the c's are scalar.
- **The span of a set of vectors from a vector space V is a subspace of V.**
- Every **subspace** is a vector space in its own right.
- Every vector space V and every subspace of V must contain the **additive identity** (usually called the zero vector).
- A **basis** for a vector space V is a linearly independent spanning set.
  - There can be many bases for a vector space.
  - Each basis for a particular vector space contains the same number of vectors.
  - SO… associated with each vector space V is the idea of dimension, denoted **dim(V)**.
    - The **dimension** of a vector space V is the number of vectors in a basis for V.
    - If **dim(V) = k**, then the **largest** subset of vectors of V that can be LI has exactly k vectors.
      - Also every spanning set for the vector space V must have at **least** k vectors. (So the smallest spanning set for a vector space is really a basis.)
      - Every spanning set of V has k or more vectors and in fact every spanning set must contain a basis.
- Associated with every matrix **A** are three subspaces.
  - The **row space of A**, denoted **row(A)**; it is the span of the rows of **A**.
    - The nonzero rows of RREF(**A**) are a basis for row(**A**).
    - Dim(row(A)) = number of nonzero rows in the RREF(A).
  - The **column space of A**, denoted **col(A)**; it is the span of the columns of **A**.
    - The nonzero columns of $(\text{RREF}(\mathbf{A}^T))^T$ are a basis for col(**A**).
    - Dim(col(**A**)) = number of nonzero columns in the $(\text{RREF}(\mathbf{A}^T))^T$.
  - It happens that dim(row(**A**)) =dim(col(**A**)). SO… associated with matrix **A** is the idea of **rank**, rank(**A**) = dim(row(**A**)) =dim(col(**A**)).
  - The **null space of A**. denoted **ns(A)**; it is the set of all solutions of the homogeneous linear system **Ax = 0**.
    - The dim(ns(**A**)) = the number of arbitrary constant in the general solution of the **Ax = 0**. (dim(ns(**A**)) is often called the nullity of **A**.)
  - It can be shown that for an m × n matrix **A** that **rank(A) + dim(ns(A)) = n**, the number of columns of **A**.
    - The general solution to the consistent nonhomogeneous linear system **Ax = b** , **b ≠ 0** can be represented by $\mathbf{x_h} + \mathbf{x_p}$ where $\mathbf{x_h}$ is the solution to **Ax = 0** and $\mathbf{x_p}$ is a particular solution to Ax = b. (By particular solution we mean a solution that contains no arbitrary constants; it contain only numerical values.)
- Rank is linked to a square matrix being singular or nonsingular.
  - n × n matrix **A** is nonsingular if and only if rank(**A**) = n.
  - n × n matrix **A** is has rank(**A**) = n if and only if det(**A**) ≠ 0.
  - n × n matrix **A** is singular if and only if rank(**A**) < n.
- Ranks is linked to homogeneous systems that have nontrivial solutions; for n × n matrix **A**, the homogeneous linear system **Ax = 0** has a nontrivial solution if and only if rank(**A**) < n.