Closure

Let S be a set on which operations of addition and scalar multiplication are defined. For example, the members of S could be

- Vectors from R^2, R^3, or in general R^n
- Matrices of size 2 × 2, 3 × 2, 10 × 6, or in general m × n; the symbol M_{mn} represents the set of m × n matrices, so M_{22}, is the set of 2 × 2 matrices, etc.
- Functions like polynomials; the symbol P_n represents the set of polynomial of degree n or less, so P_2 is the set of polynomials of degree 2 or less which consists of all quadratics $ax^2 + bx + c$. (Since the coefficient a could be zero, P_2 contains all first degree polynomials, and since a and b could be zero, P_2 contains all constant functions (polynomials of degree zero).
- Other sets of functions like continuous functions, or differentiable functions.
- Other types of quantities which would be carefully described.

Definitions

1. A set is **closed under addition** if when we add any two members of S the result is also a member of S.
2. A set is **closed under scalar multiplication** if when we take any scalar times a member of S the result is also a member of S. (The scalars are often the real numbers, but they could be the complex numbers if so specified.)

Examples of sets closed under both addition and scalar multiplication.

- R^n is closed when we use the standard definitions of addition of vectors and scalar multiplication of vectors.
- M_{mn} is closed when we use the standard definitions of addition of matrices and scalar multiplication of matrices.
- P_n is closed when we use the standard definitions of addition of polynomials and scalar multiplication of polynomials.

WARNING: If we change the definitions of addition or scalar multiplication then the sets in the preceding examples may not be closed. We would need to carefully check the definitions.

Example 1. Let $S = R^3$ with the following definitions of addition and scalar multiplication. (We use the symbol $\oplus$ to denote addition of vectors and the symbol $\odot$ to denote scalar multiplication of a vector.)

For **u** and **v** in R^3 $\mathbf{u} \oplus \mathbf{v} = (u_1,u_2,u_3) \oplus (v_1,v_2,v_3) = (u_1+v_1,\ 0,\ u_2+v_3)$.

For c any real scalar and **u** in R^3 $c \odot \mathbf{u} = c \odot (u_1,u_2,u_3) = (0,0,u_1+u_2)$

Is S closed under the addition operation $\oplus$? Answer, YES since the result is a vector in R^3.
Is S closed under the scalar multiplication operation $\odot$? Answer, YES since the result is a vector in R^3.

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Example 2. Let S = M_{22} with the following definitions of addition and scalar multiplication. (We use the symbol $\oplus$ to denote addition of vectors and the symbol $\odot$ to denote scalar multiplication of a vector.)

For **A** and **B** in M_{22} $\mathbf{A} \oplus \mathbf{B} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \oplus \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_2 b_2 \\ a_3 b_3 & a_4 b_4 \end{bmatrix}$.

For c any real scalar and **A** in M_{22} $c \odot \mathbf{A} = c \odot \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = c a_1 a_4$.

Is S closed under the addition operation $\oplus$? Answer, YES since the result is in M_{22}.
Is S closed under the scalar multiplication operation $\odot$? Answer, NO since the result is not in R^3.

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As Examples 1 and 2 illustrate, even if we alter the standard definitions of addition and scalar multiplication checking if a set is closed under the operations we merely apply the definitions.

Another situation that arises when the set S is a subset of a familiar set like R^n, M_{mn}, or P_n where the operations are the standard additions and scalar multiplication. In such cases, we must carefully check to determine if the result of the addition and scalar multiplication are members of the set S, not just in the more familiar larger set. We illustrate this in the following examples.

Example 3. Let S be the set of all vectors **v** in R^2 of the form $\mathbf{v} = \begin{bmatrix} r \\ 0 \end{bmatrix}$ where r is any real number.

Is S closed under the standard definition of addition in R^2?

If **v** and **w** are in S with $\mathbf{v} = \begin{bmatrix} r \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} t \\ 0 \end{bmatrix}$, t any real number,

then $\mathbf{v} + \mathbf{w} = \begin{bmatrix} r+t \\ 0 \end{bmatrix}$ which is in set S. Thus S is closed under addition.

Is S closed under the standard definition of scalar multiplication in R^2?

If **v** is in S with $\mathbf{v} = \begin{bmatrix} r \\ 0 \end{bmatrix}$ and c is any real number, then $c\mathbf{v} = \begin{bmatrix} cr \\ 0 \end{bmatrix}$ which is in set S.

Thus S is closed under scalar multiplication.

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Example 4. Let S be the set of all vectors **v** in R^4 with positive entries.

Is S closed under the standard definition of addition in R^4?

Yes, since adding two such vectors gives another vector with all positive entries.

Is S closed under the standard definition of scalar multiplication in R^4?

No, for scalar c negative or zero, c**v** will not be in the set S.

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CLOSED SETS EXERCISES

Exercises (Just answering yes or no is not enough in the following exercises. You must give a valid reason for your answer.)

1. Let S be the set of all vectors **v** in R^3 of the form $\mathbf{v} = \begin{bmatrix} r \\ 0 \\ 1 \end{bmatrix}$ where r is any real number.
   (a) Is S closed under the standard operation of addition in R^3?
   (b) Is S closed under the standard operation of scalar multiplication in R^3?

2. Let S be the set of all symmetric 2 × 2 matrices.
   (a) Is S closed under the standard operation of addition in M_{22}?
   (b) Is S closed under the standard operation of scalar multiplication in M_{22}?

3. Let S be the vectors in R^4 of the form (r, s, r+s, 0) where rand s are any real numbers.
   (a) Is S closed under the standard operation of addition in R^4?
   (b) Is S closed under the standard operation of scalar multiplication in R^4?

4. Let S be the set of all 3 × 3 nonsingular matrices.
   (a) Is S closed under the standard operation of addition in M_{33}?
   (b) Is S closed under the standard operation of scalar multiplication in M_{33}?

5. Let **A** be singular 10 × 10 matrix and S be the set of all solutions to **Ax = 0**.
   (a) Is S closed under the standard operation of addition in R^{10}?
   (b) Is S closed under the standard operation of scalar multiplication in R^{10}?