Determinants by Expansion

Here we develop a procedure for computing the determinant of an n × n matrix as a linear combination of determinants of smaller matrices. We call this method **determinants by expansion**.

The expression for the determinant of a 3 × 3 matrix can be written as an expansion involving determinants of 2 × 2 submatrices and the entries from the first row. We have (verify by using the 2 × 2 device and simplifying)

$$\mathbf{det}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a * \mathbf{det}\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b * \mathbf{det}\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c * \mathbf{det}\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right).$$

Note that

$\begin{bmatrix} e & f \\ h & i \end{bmatrix}$ is the submatrix obtained by omitting the first row and first column,

$\begin{bmatrix} d & f \\ g & i \end{bmatrix}$ is the submatrix obtained by omitting the first row and second column,

and

$\begin{bmatrix} d & e \\ g & h \end{bmatrix}$ is the submatrix obtained by omitting the first row and third column.

In addition, we can incorporate the row and column numbers of the rows and columns omitted by writing the expression in the form

$$\mathbf{det}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = (-1)^{1+1} a * \mathbf{det}\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) + (-1)^{1+2} b * \mathbf{det}\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + (-1)^{1+3} c * \mathbf{det}\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

We call this expression **the expansion of the determinant about the first row**. In fact, we can use any row or column for the expansion with appropriate powers of (-1) multiplying the entries and submatrices selected by omitting a row and column.

Example: To compute the determinant of matrix $\mathbf{A} = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & 5 \\ -6 & 2 & 7 \end{bmatrix}$ by expansion about the first row we will form a linear combination of determinants of 2 × 2 submatrices that have coefficients from entries of row 1 multiplied by (1) raised to a power which is the sum of the row & column location of the entry. We first identify the 2 × 2 submatices involved. We form these submatrices by omitting row 1 and successive columns:

omitting the first row and first column ➔ $\begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$

omitting the first row and second column ➔ $\begin{bmatrix} 0 & 5 \\ -6 & 7 \end{bmatrix}$

omitting the first row and third column ➔ $\begin{bmatrix} 0 & 1 \\ -6 & 2 \end{bmatrix}$

Next we form an appropriate linear combination of the determinants of these submatrices:

$$\mathbf{det(A)} = (-1)^{1+1}3\,\mathbf{det}\left(\begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}\right) + (-1)^{1+2}\,\mathbf{(}-2\mathbf{)}\,\mathbf{det}\left(\begin{bmatrix} 0 & 5 \\ -6 & 7 \end{bmatrix}\right) + (-1)^{1+3}\,4\,\mathbf{det}\left(\begin{bmatrix} 0 & 1 \\ -6 & 2 \end{bmatrix}\right)$$

Compute the determinants of the 2 × 2 submatrices using our special device. Then we get

$$\mathbf{det(A)} = (-1)^{1+1}3(-3) + (-1)^{1+2}\,\mathbf{(}-2\mathbf{)}(30) + (-1)^{1+3}\,4(6) = -9 + 60 + 24 = 75$$

+++

The definition we give for the determinant of an n ×n matrix is an expansion about the first row following the pattern of the 3 × 3 case as given above. However, the expression involves determinants of (n -1) × (n -1) matrices, hence is recursive.

<u>Definition</u> The determinant of an n × n matrix **A** = $[a_{ij}]$, denoted **det(A)**, is given by the expression

$$\mathbf{det(A)} = (-1)^{1+1}a_{11}\,\mathbf{det(A_{11})} + (-1)^{1+2}a_{12}\,\mathbf{det(A_{12})} + (-1)^{1+3}a_{13}\,\mathbf{det(A_{13})} + \cdots + (-1)^{1+\mathbf{n}}a_{1n}\,\mathbf{det(A_{1n})}$$

where $\mathbf{A}_{1k}$ is the submatrix obtained from **A** by omitting the first row and k^{th} column. (We call this **expansion about the first row**.)

We state without verification that expansion about any row or column will yield the same value.

Example: To compute the determinant of matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & 5 & 3 \\ 0 & 6 & -2 & 1 \\ 4 & 0 & 2 & 0 \\ 5 & 7 & 0 & -3 \end{bmatrix}$ by expansion we will use expansion about the third row. The reason we choose the third row is that two of its entries are zero, hence the computation of det(A) as a linear combination of determininants of 3 × 3 submatrices will involve finding the determinant of only two 3 × 3 submatrices instead of four such computations had we used row 1. We first identify the 3 × 3 submatices involved. We form these submatrices by omitting row 3 and successive columns:

omitting the third row and the first column ➔ $\begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 1 \\ 7 & 0 & -3 \end{bmatrix}$

omitting the third row and the second column ➔ $\begin{bmatrix} 2 & 5 & 3 \\ 0 & -2 & 1 \\ 5 & 0 & -3 \end{bmatrix}$

omitting the third row and the third column ➔ $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 6 & 1 \\ 5 & 7 & -3 \end{bmatrix}$

omitting the third row and the fourth column ➔ $\begin{bmatrix} 2 & -1 & 5 \\ 0 & 6 & -2 \\ 5 & 7 & 0 \end{bmatrix}$

Next we form an appropriate linear combination of the determinants of these submatrices:

$$\mathbf{det(A)} = (-1)^{3+1}\,4\,\mathbf{det}\left(\begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 1 \\ 7 & 0 & -3 \end{bmatrix}\right) + (-1)^{3+2}\,0\,\mathbf{det}\left(\begin{bmatrix} 2 & 5 & 3 \\ 0 & -2 & 1 \\ 5 & 0 & -3 \end{bmatrix}\right)$$
$$+ (-1)^{3+3}\,2\,\mathbf{det}\left(\begin{bmatrix} 2 & -1 & 3 \\ 0 & 6 & 1 \\ 5 & 7 & -3 \end{bmatrix}\right) + (-1)^{3+4}\,0\,\mathbf{det}\left(\begin{bmatrix} 2 & -1 & 5 \\ 0 & 6 & -2 \\ 5 & 7 & 0 \end{bmatrix}\right)$$

Simplifying this linear combination using that $a_{3,2} = a_{3,4} = 0$, we have that

$$\mathbf{det(A)} = (-1)^{3+1}\, 4\, \mathbf{det}\left(\begin{bmatrix} -1 & 5 & 3 \\ 6 & -2 & 1 \\ 7 & 0 & -3 \end{bmatrix}\right) + (-1)^{3+3}\, 2\, \mathbf{det}\left(\begin{bmatrix} 2 & -1 & 3 \\ 0 & 6 & 1 \\ 5 & 7 & -3 \end{bmatrix}\right)$$

Computing the determinants of the 3 × 3 submatrices we have that

$$\mathbf{det(A)} = (-1)^{3+1}\, 4\mathbf{(}161\mathbf{)} + (-1)^{3+3}\, 2\mathbf{(}-145\mathbf{)} = 354\,.$$

++

The method we have called determinants by expansion is sometimes called Cofactor Expansion or Laplace Expansion by Cofactors. This method is also referred to as a recursive method for determinant calculation.

Computing the Determinant using Row Operations

Computing determinant of an n × n matrix for large n using expansion is quite tedious. A more efficient technique involves the use of row operations. However, row operations can change the value of a determinant. From Table 1 we have

$$\mathbf{det(A}_{kR_i}\mathbf{)} = k\,\mathbf{det(A)}$$
$$\mathbf{det(A}_{R_i \leftrightarrow R_j}\mathbf{)} = -\mathbf{det(A)},\ i \neq j$$
$$\mathbf{det(A}_{kR_i + R_j}\mathbf{)} = \mathbf{det(A)}$$

and **det(**upper triangular matrix**)** = product of its diagonal entries. Using these four properties will formulate a procedure for computing the determinant of any square matrix by using row operations to reduce it to upper triangular form and keeping track of how the row operations employed effect the determinant.

For convenience we adopt the following notation:

- Our original matrix will be denoted by **A**.
- When a row operation is applied to a matrix we append a subscript indicating the row operation to the name of that matrix. For example

  $$\mathbf{B}_{\mathbf{R_i} \leftrightarrow \mathbf{R_j}}$$

  indicates that we have interchanged rows i and j of matrix **B**.

Here we give a brief explanation of the effect of row operations on the determinant.

- If we multiply a row of **A** by k ≠ 0, then the determinant of resulting matrix is k times the determinant of **A**. Thus the determinant of **A** is 1/k times the determinant of matrix $\mathbf{A_{kR_i}}$. That is,

$$\mathbf{det(A)} = \frac{1}{\mathbf{k}}\mathbf{det(A_{kR_i})}\ \mathbf{k} \neq 0\,.$$

- If we interchange a pair of rows of **A**, then the determinant of the resulting matrix is the negative of the determinant of **A**. Thus the determinant of **A** is (-1) times the determinant of $\mathbf{A_{R_i \leftrightarrow R_j}}$. That is,

$$\mathbf{det(A)} = -\mathbf{det(A_{R_i \leftrightarrow R_j})}$$

- If we add a multiple of one row of **A** to another row, then the determinant doesn't change. That is,

$$\mathbf{det(A)} = \mathbf{det(A_{kR_i + R_j})}$$

Our ***computational strategy*** for computing det(**A**) is to apply row operations to matrix **A** to produce an upper triangular matrix while keeping track of the effect the row operations have on the determinant. Since at each step we want an expression for det(**A**) we will make use of the following relationships.

$$\mathbf{det(A)} = \frac{1}{\mathbf{k}}\mathbf{det(A_{kR_i})}\ \mathbf{k} \neq 0$$

$$\mathbf{det(A)} = -\mathbf{det(A_{R_i \leftrightarrow R_j})}$$

$$\mathbf{det(A)} = \mathbf{det(A_{kR_i + R_j})}$$

Example: Compute $\mathbf{det(A)} = \mathbf{det}\left(\begin{bmatrix} 2 & 4 & 0 & -6 \\ 1 & 0 & 2 & 3 \\ -3 & 5 & -1 & 0 \\ 0 & -2 & -7 & 1 \end{bmatrix}\right)$ by using row operations.

To get a 1 into the first pivot position interchange rows 1 and 2. Let $\mathbf{B} = \mathbf{A_{R_1 \leftrightarrow R_2}}$ then

$$\mathbf{det(A)} = -\mathbf{det(B)} = -\mathbf{det}\left(\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 4 & 0 & -6 \\ -3 & 5 & -1 & 0 \\ 0 & -2 & -7 & 1 \end{bmatrix}\right).$$

Next eliminate below the first pivot in **B** using row operations $-2\mathbf{R}_1+\mathbf{R}_2$ and $3\mathbf{R}_1+\mathbf{R}_3$. Call the resulting matrix **C**. We have $\mathbf{C}=\mathbf{B}_{\substack{-2\mathbf{R}_1+\mathbf{R}_2\\ 3\mathbf{R}_1+\mathbf{R}_3}}$ and

$\mathbf{det(C)}=\mathbf{det(B)}$ thus

$$\mathbf{det(A)}=-\mathbf{det(B)}=-\mathbf{det(C)}=-\mathbf{det}\left(\begin{bmatrix}1 & 0 & 2 & 3\\ 0 & 4 & -4 & -12\\ 0 & 5 & 5 & 9\\ 0 & -2 & -7 & 1\end{bmatrix}\right).$$

Now to get a 1 in the second pivot position we use row operation $(1/4)\mathbf{R}_2$ on matrix **C**. Call the resulting **D**. We have $\mathbf{D}=\mathbf{C}_{(1/4)\mathbf{R}_2}$ and $\mathbf{det(D)}=(1/4)\,\mathbf{det(C)}$ or equivalently $\mathbf{det(C)}=4\,\mathbf{det(D)}$. Hence we get

$$\mathbf{det(A)}=-\mathbf{det(C)}=-4\,\mathbf{det(D)}=-4\,\mathbf{det}\left(\begin{bmatrix}1 & 0 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & 5 & 5 & 9\\ 0 & -2 & -7 & 1\end{bmatrix}\right).$$

Next eliminate below the second pivot using row operations $-5\mathbf{R}_2+\mathbf{R}_3$ and $2\mathbf{R}_2+\mathbf{R}_4$. Call the resulting matrix **E**. We have $\mathbf{E}=\mathbf{D}_{\substack{-5\mathbf{R}_2+\mathbf{R}_3\\ 2\mathbf{R}_2+\mathbf{R}_4}}$ and $\mathbf{det(E)}=\mathbf{det(D)}$. It follows

that

$$\mathbf{det(A)}=-4\,\mathbf{det(D)}=-4\,\mathbf{det(E)}=-4\,\mathbf{det}\left(\begin{bmatrix}1 & 0 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & 0 & 10 & 24\\ 0 & 0 & -9 & -5\end{bmatrix}\right).$$

Using one more row operation we will have a matrix row equivalent to **A** that is upper triangular. Perform row operation $(9/10)\mathbf{R}_3+\mathbf{R}_4$ on matrix E. Call the resulting matrix **F** and we have $\mathbf{det(F)}=\mathbf{det(E)}$ so

$$\mathbf{det(A)}=-4\,\mathbf{det(E)}=-4\,\mathbf{det(F)}=-4\,\mathbf{det}\left(\begin{bmatrix}1 & 0 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & 0 & 10 & 24\\ 0 & 0 & 0 & 83/5\end{bmatrix}\right).$$

Since matrix **F** is upper triangular we can compute its determinant as the product of its diagonal entries and we have

$$\mathbf{det(A)}=-4\,\mathbf{det(F)}=-4(1)(1)(10)(83/5)=-664\,.$$

+++