Diagonalization

Some square matrices **A** are intimately related to a diagonal matrix **D** which has the same eigenvalues as **A**. Here we define and discuss such matrices.

DEFINITION:
A matrix **B** is said to be **similar** to a matrix **A** if there is a nonsingular matrix **P** such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Properties of similar matrices:

- **A** is similar to itself. (Just take **P** = **I**.)
- If **B** is similar to **A**, then **A** is similar to **B**.

  ($\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is equivalent to $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$.)
- If **A** is similar to **B** and **B** is similar to **C**, then **A** is similar to **C**.

($\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ and $\mathbf{B} = \mathbf{Q}\mathbf{C}\mathbf{Q}^{-1}$, then $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1} = \mathbf{P}\mathbf{Q}\mathbf{C}\mathbf{Q}^{-1}\mathbf{P}^{-1} = (\mathbf{PQ})\mathbf{C}(\mathbf{PQ})^{-1}$.)

There is a special class of similar matrices which we define next.

DEFINITION:
A matrix **A** is called **diagonalizable** if it is similar to a diagonal matrix. In this case we say that **A** can be **diagonalized**.

Properties of Similar and Diagonalizable matrices.

1. Similar matrices have the same eigenvalues.

2. If **A** is diagonalizable then it is similar to a diagonal matrix **D**. The diagonal entries of **D** are the eigenvalues of **A**.

3. An $n \times n$ matrix **A** is diagonalizable if and only if it has n linearly independent eigenvectors. In this case the matrix **P** so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, has columns which are the linearly independent eigenvectors of **A**.

4. If all the eigenvalues of $n \times n$ matrix **A** are distinct then, **A** is diagonalizable.

5. A matrix which is not diagonalizable is called **defective**.

Computational Examples

Example 1. Matrix $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{2} \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 4$ with corresponding eigenvectors $\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \end{bmatrix}$. Show that **A** is diagonalizable.

We form matrix $\mathbf{P} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix}$, compute its inverse to get $\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{1/3} & \mathbf{1/3} \\ \mathbf{2/3} & \mathbf{-1/3} \end{bmatrix}$, and now compute $\mathbf{P^{-1}AP} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} \end{bmatrix}$.

Example 2. Determine if $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$ is diagonalizable.

Note that A is upper triangular so its eigenvalues are its diagonal entries. We know that A has at least two linearly independent eigenvectors since one eigenvalue is 2 and another is 3. We need to determine the number of linearly independent eigenvectors corresponding to eigenvalue 2. So we solve homogeneous system $\mathbf{(2I - A)x = 0}$. We get augmented matrix $\left[\begin{array}{ccc|c} \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \end{array}\right]$ which has RREF $\left[\begin{array}{ccc|c} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$. There is only 1 arbitrary constant so there is only one linearly independent eigenvector corresponding to eigenvalue 2. Hence **A** is not diagonalizable. It is defective.

Example 3. Determine which of the following matrices are diagonalizable.

(a) $\mathbf{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} \\ \mathbf{3} & \mathbf{3} \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} & \mathbf{3} \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$