Eigen Exercises

Key
(NAME)

1. If matrix A is 2 by 2 and has eigenvalues $\lambda = 1$ and $\lambda = -2$, what are the eigenvalues of A^3?
   Answer: $1^3, (-2)^3 \Rightarrow 1, -8$

2. The characteristic polynomial of matrix A is $\lambda^4 - 3\lambda^2 - 4$. $\rightarrow (\lambda^2+1)(\lambda^2-4)$
   (a) What size is matrix A? 4x4
   (b) What are the eigenvalues of A? $2, -2, \pm i$
   (c) How many linearly independent eigenvectors does matrix A have? 4
   (because the eigenvalues are distinct.)

3. The characteristic polynomial of matrix A is $\lambda^3 + 4\lambda^2 + 4\lambda$. $\rightarrow \lambda(\lambda^2+4\lambda+4) = \lambda(\lambda+2)^2$
   (a) What size is matrix A? 3x3
   (b) What are the eigenvalues of A? $0, -2, -2$
   (c) Is it guaranteed that A has 3 linearly independent eigenvectors?
   Circle the appropriate response: YES NO
   Briefly explain why your answer is correct: We have a repeated eigenvalue

4. Find a basis for the eigenspace of $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ corresponding to eigenvalue 4.

   Basis: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ What is t he dimension of this eigenspace? 1

5. Find a basis for the eigenspace of $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ corresponding to eigenvalue 4.

   Basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ What is t he dimension of this eigenspace? 2

6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) $\lambda = 3$ is an eigenvalue of A. Find a corresponding eigenvector. Answer: $r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, r \neq 0$
(b) Without doing any computations name an eigenvalue of A different from $\lambda = 3$.
   Answer: 0 Give a reason that you answer is correct: Matrix is singular

7. The eigenvalues of A are 2, -3, 6, and 1. What is det(A)? Answer: $(2)(-3)(6)(1) = -36$

8. Let $A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$. One eigenvalue of A is $\lambda = 4$.

   (a) Find an eigenvector of A corresponding to $\lambda = 4$. Answer: $r\begin{bmatrix} -2 \\ 1 \end{bmatrix}, r \neq 0$

   (b) Find an eigenvector of A^T corresponding to $\lambda = 4$. Answer: $r\begin{bmatrix} -1 \\ 1 \end{bmatrix}, r \neq 0$

9. Construct a matrix whose characteristic polynomial is $\lambda^3 - 3\lambda^2 + 2\lambda$. $\rightarrow \lambda(\lambda^2 - 3\lambda + 2)$

$\lambda(\lambda-1)(\lambda-2)$

so eigenvalues are $\lambda = 0, 1, 2$

The eigenvalues are distinct so the matrix is similar to a diagonal matrix

ONE is

Answer: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. If square matrix A has a row of zeros, what is one eigenvalue of A? Answer: 0

Give a reason that you answer is correct: Matrix is singular

11. If square matrix A has a column of zeros, what is one eigenvalue of A? Answer: 0

Give a reason that you answer is correct: Matrix is singular

12. A is a square matrix such that the sum of the entries in each row is the number k. Prove that k is an eigenvalue of A.

Put your Proof here: $A\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

this product just 'sums' the entries in each row.

13. A is a square matrix such that the sum of the entries in each column is the number r. Prove that r is an eigenvalue of A.

Put your Proof here: Recall that A and A^T have the same eigenvalues.

So $A^T\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ see # 12

14. Let S be the set of all 3 by 3 diagonal matrices that have eigenvalue λ = 7. Is S a subspace of M_{33}? Explain.

Every matrix in S has a 7 on the diagonal.

If A and B are in S does A+B have a 7 on the diagonal.

Not necessarily; for example $\begin{bmatrix} 7 & & \\ & 1 & \\ & & 2 \end{bmatrix} + \begin{bmatrix} 5 & & \\ & 7 & \\ & & 9 \end{bmatrix} = \begin{bmatrix} 12 & & \\ & 8 & \\ & & 11 \end{bmatrix} \Rightarrow$ Not a subspace

Also, if A is in S, $0A = 0$ which is not in S.

15. Let A be a 3 by 3 diagonal matrix such that $\lim_{k\to\infty} A^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Describe the set of eigenvalues of A.

Answer: Let $A = \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix}$, then

$A^k = \begin{bmatrix} r^k & 0 & 0 \\ 0 & s^k & 0 \\ 0 & 0 & t^k \end{bmatrix}$ As $k \to \infty$

$\left.\begin{array}{l} \lim_{k\to\infty} r^k = 0 \\ \lim_{k\to\infty} s^k = 0 \\ \lim_{k\to\infty} t^k = 0 \end{array}\right\} \Rightarrow |r|, |s|, |t|$ are all less than 1