Section 8.2 Solutions to assigned Exercises

2. Call the matrix A. The matrix is lower triangular so its eigenvalues are $\lambda = 1$ and 1.

Then $(\lambda I - A)x = \left(1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ has the RREF of the augmented matrix given by $\left[\begin{array}{cc|c}1 & 1/2 & 0\\0 & 0 & 0\end{array}\right]$. Thus there is one arbitrary constant so one linear independent eigenvector corresponding to the repeated eigenvalue. So A is not diagonalizable.

3. Call the matrix A. First find its eigenvalues:

$$\mathbf{det}(\lambda I - A) = \mathbf{det}\left(\begin{bmatrix}\lambda-1 & -1 & 2\\-4 & \lambda & -4\\-1 & 1 & \lambda-4\end{bmatrix}\right) = \lambda^3 - 5\lambda^2 + 6\lambda = \lambda(\lambda-2)(\lambda-3) \Rightarrow$$

eigenvalues are 0, 2, 3. Since the eigenvalues are distinct it is diagonalizable.

4. Note that the matrix is upper triangular, so its eigenvalues are the diagonal entries. Since the eigenvalues are distinct it is diagonalizable.

6. Call the matrix A. First find its eigenvalues:

$$\mathbf{det}(\lambda I - A) = \mathbf{det}\left(\begin{bmatrix}\lambda+2 & -2\\-5 & \lambda-1\end{bmatrix}\right) = \lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3) \Rightarrow \text{eigenvalues}$$

are -4 and 3. Since the eigenvalues are distinct it is diagonalizable.

9. One such matrix is $A = \begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}2 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix} = \begin{bmatrix}-4/3 & -5/3\\-10/3 & 1/3\end{bmatrix}$.

11. Call the matrix A. We first find its eigenvalues from the characteristic polynomial, then find the corresponding eigenvectors.

$$\mathbf{det}(\lambda I - A) = \mathbf{det}\left(\begin{bmatrix}\lambda-4 & -2 & -3\\-2 & \lambda-1 & -2\\1 & 2 & \lambda\end{bmatrix}\right) = \lambda^3 - 5\lambda^2 + 7\lambda - 3 = (\lambda-1)^2(\lambda-3)$$

Then the eigenvalues are 1, 1, 3. First find eigenvectors corresponding to eigenvalue 1. Homogeneous system (1I – A)x = 0 has augmented matrix $\left[\begin{array}{ccc|c}-3 & -2 & -3 & 0\\-2 & 0 & -2 & 0\\1 & 2 & 1 & 0\end{array}\right]$ and its RREF is $\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$. We see there will be just one arbitrary constant so only one linearly independent eigenvector corresponding to repeated eigenvalue 1. So A is not diagonalizable.

12. Call the matrix A. We first find its eigenvalues from the characteristic polynomial, then find the corresponding eigenvectors.

$$\mathbf{det}(\lambda I - A) = \mathbf{det}\left(\begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda-3 \end{bmatrix}\right) = \lambda^3 - 5\lambda^2 + 7\lambda - 3 = (\lambda-1)^2(\lambda-3)$$

Then the eigenvalues are 1, 1, 3. First find eigenvectors corresponding to eigenvalue 1. Homogeneous system (1I – A)x = 0 has augmented matrix $\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$ and its RREF is $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. We see there will be two arbitrary constants so we have two linearly independent eigenvectors corresponding to repeated eigenvalue 1. We find that a pair of linearly independent eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Next we find an eigenvector corresponding to eigenvalue 3. Homogeneous system (3I – A)x = 0 has augmented matrix $\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$ and its RREF is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. So an eigenvector corresponding to eigenvalue 3 is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. We can take $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

16. Call the matrix A. We first find its eigenvalues from the characteristic polynomial, then find the corresponding eigenvectors.

$$\mathbf{det}(\lambda I - A) = \mathbf{det}\left(\begin{bmatrix} \lambda-3 & 0 & 0 \\ -1 & \lambda-3 & -1 \\ 0 & 0 & \lambda-1 \end{bmatrix}\right) = \lambda^3 - 7\lambda^2 + 15\lambda - 9 = (\lambda-3)^2(\lambda-1)$$

Then the eigenvalues are 1, 3, 3. First find eigenvectors corresponding to eigenvalue 3. Homogeneous system (3I – A)x = 0 has augmented matrix $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ and its RREF is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. We see there will be just one arbitrary constant so only one linearly independent eigenvector corresponding to repeated eigenvalue 3. So A is not diagonalizable.

23. $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$.

41. $(\lambda I - A)x = \left(8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 7 \\ 0 & 8 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has the RREF of the augmented matrix given by $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Thus there is one arbitrary constant so one linear independent eigenvector corresponding to repeated eigenvalue 8. So the matrix is defective.

42. $(\lambda I - A)x = \left(3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has the RREF of the augmented matrix given by $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus there are two arbitrary constants so two linear independent eigenvectors corresponding to repeated eigenvalue 3. So the matrix is not defective.

T9. If matrix **B** is similar to a matrix **A** if there is a nonsingular matrix **P** such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}.$$

Then

$$\mathbf{det}(B) = \mathbf{det}\left(P^{-1}AP\right) = \mathbf{det}\left(P^{-1}\right)\mathbf{det}(A)\,\mathbf{det}(P) = \frac{1}{\mathbf{det}(P)}\mathbf{det}(A)\,\mathbf{det}(P) = \mathbf{det}(A).$$