Sec 3.1 P193 Use 2x2 & 3x3 "Devices"

#5(a) $\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 - (3)(-1) = \boxed{7}$

(c) $\left|\begin{array}{ccc|cc} 4 & 2 & 0 & 4 & 2 \\ 0 & -2 & 5 & 0 & -2 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right.$

$(-24) + 0 + 0 - [0 + 0 + 0]$
$= \boxed{-24}$

#6 (a) $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 6 - (4)(1) = \boxed{2}$

(b) $\left|\begin{array}{ccc|cc} 0 & 0 & -2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 4 & 0 & 0 & 4 & 0 \end{array}\right.$

$0 + 0 + 0 - [-24 + 0 + 0] = \boxed{24}$

(c) $\left|\begin{array}{ccc|cc} 3 & 4 & 2 & 3 & 4 \\ 2 & 5 & 0 & 2 & 5 \\ 3 & 0 & 0 & 3 & 0 \end{array}\right. = 0 + 0 + 0 - [30 + 0 + 0] = -30$

#11 (a) $\det\left(\begin{bmatrix} \lambda-1 & 2 \\ 3 & \lambda-2 \end{bmatrix}\right) = (\lambda-1)(\lambda-2) - 6 = \lambda^2 - 3\lambda + 2 - 6$
$= \boxed{\lambda^2 - 3\lambda - 4}$

(b) $\det(\lambda I_2 - A| = \det\left(\begin{bmatrix} \lambda-4 & -2 \\ 1 & \lambda-1 \end{bmatrix}\right) = (\lambda-4)(\lambda-1) + 2$

#12 (a) $\det\left(\begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-2 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}\right) = (\lambda-1)(\lambda-2)(\lambda-3)$
$= (\lambda^2 - 3\lambda + 2)(\lambda-3) = \lambda^3 - 3\lambda^2 + 2\lambda$
$\quad - 3\lambda^2 + 9\lambda - 6$
$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$

Note matrix is diagonal

(b) $\det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda+1 & 0 & -1 \\ 2 & \lambda & 1 \\ 0 & 0 & \lambda-1 \end{bmatrix}\right)$

Use 3x3 trick $\left[\begin{array}{ccc|cc} \lambda+1 & 0 & -1 & \lambda+1 & 0 \\ 2 & \lambda & 1 & 2 & \lambda \\ 0 & 0 & \lambda-1 & 0 & 0 \end{array}\right]$

$(\lambda+1)\lambda(\lambda-1) + 0 + 0 - 0 - 0 - 0 = (\lambda^2 - 1)\lambda = \lambda^3 - \lambda$

$$= \lambda^2 - 5\lambda + 4 + 2 = \boxed{\lambda^2 - 5\lambda + 6}$$

#13 (a) $\lambda^2 - 3\lambda - 4 = 0 \Rightarrow (\lambda-4)(\lambda+1) = 0 \Rightarrow \boxed{\lambda = 4,\ \lambda = -1}$

(b) $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-3)(\lambda-2) = 0 \Rightarrow \boxed{\lambda = 3,\ \lambda = 2}$

#14 (a) we have from #12 $\lambda^3 - 6\lambda^2 + 11 - 6 = (\lambda-1)(\lambda-2)(\lambda-3) = 0$ before we expanded

So $\lambda = 1, 2, \text{ or } 3$

(b) We have $\lambda^3 - \lambda = \lambda(\lambda^2 - 1) = 0 \Rightarrow \lambda = 0, 1, -1$

17 (a)
$$\left|\begin{array}{ccc|cc} 4 & -3 & 5 & 4 & -3 \\ 5 & 2 & 0 & 5 & 2 \\ 2 & 0 & 4 & 2 & 0 \end{array}\right| = 32 + 0 + 0 - [20 + 0 - 60]$$
$$= 32 - [-40] = \boxed{72}$$

#18 (b)
$$\left|\begin{array}{ccc|cc} 4 & 1 & 3 & 4 & 1 \\ 2 & 3 & 0 & 2 & 3 \\ 1 & 3 & 2 & 1 & 3 \end{array}\right| = 24 + 0 + 18 - [9 + 0 + 4] = 42 - 13 = \boxed{29}$$

(c)
$$\left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 \\ -3 & 1 & 2 & -3 & 1 \end{array}\right| = 2 + 0 + 6 - [-9 + 0 + 8] = 8 - (-1) = \boxed{9}$$

#20 (a) $\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}\right)$ use 3x3 trick

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 & 1 \end{array}\right]$$

We get $0 + 0 + 1 - 2 - 0 - 0 = -1$

Sec 3.1 Compute determinants via properties or rules

\# 5(c) is upper triangular to det = product of diagonal entries
so det $= 4(-2)(3) = -24$

\# 15 (a) Note there is a column of zeros so det $=0$
OR
Use that the matrix is upper triangular so det $=0(4)(-1)=0$

\# 15 (b) Matrix is upper triangular so det = product
of diagonal entries $\Rightarrow$ det $= 6(4)(-3)(2) = -144$

\# 16 (a) Matrix is lower triangular so det = product
of diagonal entries $\Rightarrow$ det $= 6(4)(-3)(2) = -144$

\# 18 (a) Matrix is lower triangular so det = product
of diagonal entries $\Rightarrow$ det $= 4(2)(-3)(5) = -120$

\# 20 (b) Matrix is lower triangular so det = product
of diagonal entries $\Rightarrow$ det $= 2(3)(4)(-5) = -120$

Section 3.1 Use the method of row operations (reduction to upper triangular form) to find the determinant.

#17 (a) Find det(A)

A =

$$\begin{matrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{matrix}$$

Use row operation Row 1 <--> Row 3 on A to get matrix B.

B =

$$\begin{matrix} 2 & 0 & 4 \\ 5 & 2 & 0 \\ 4 & -3 & 5 \end{matrix}$$

So $\det(A) = -\det(B)$

Use row operation 1/2 * Row 1 on B to get matrix C. $\Rightarrow \det(C) = \frac{1}{2}\det(B)$

C =

$$\begin{matrix} 1 & 0 & 2 \\ 5 & 2 & 0 \\ 4 & -3 & 5 \end{matrix}$$

So $\det(A) = -\det(B) = -2\det(C)$

Use row operation -5 * Row 1 + Row 2 on C to get matrix D. $\Rightarrow \det(D) = \det(C)$

D =

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 2 & -10 \\ 4 & -3 & 5 \end{matrix}$$

So $\det(A) = -2\det(C) = -2\det(D)$

Use row operation -4 * Row 1 + Row 3 on D to get matrix E. $\Rightarrow \det(E) = \det(D)$

E =

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 2 & -10 \\ 0 & -3 & -3 \end{matrix}$$

So $\det(A) = -2\det(D) = -2\det(E)$

Use row operation 0.5 * Row 2 on E to get matrix F. $\Rightarrow \det(F) = \frac{1}{2}\det(E)$

F =

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & -3 & -3 \end{matrix}$$

So $\det(A) = -2\det(E) = (-2)(2)\det(F)$

Use row operation 3 * Row 2 + Row 3 to get matrix G. $\Rightarrow \det(G) = \det(F)$

G =

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & -18 \end{matrix}$$

So $\det(A) = -4\det(G)$

$= -4(-18) = 72$

Section 3.1 Use reduction to upper triangular form

17(b) Find det(A)

A =

$$\begin{matrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{matrix}$$

Use row operation 1/2 * Row 1 on A to get matrix B $\Rightarrow \det(B) = \frac{1}{2}\det(A)$

B =

$$\begin{matrix} 1 & 0 & 1/2 & 2 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{matrix}$$

So $\det(A) = 2\det(B)$

(Multiple row ops on B)
Use row operations -3 * Row 1 + Row 2
-2 * Row 1 + Row 3
-11 * Row 1 + Row 4 on B to get matrix C.

$\Rightarrow \det(C) = \det(B)$

C =

$$\begin{matrix} 1 & 0 & 1/2 & 2 \\ 0 & 2 & -11/2 & -8 \\ 0 & 3 & -2 & -4 \\ 0 & 8 & -19/2 & -16 \end{matrix}$$

So $\det(A) = 2\det(C)$

Use row operation 1/2 * Row 2 on C to get matrix D. $\Rightarrow \det(D) = \frac{1}{2}\det(C)$

D =

$$\begin{matrix} 1 & 0 & 1/2 & 2 \\ 0 & 1 & -11/4 & -4 \\ 0 & 3 & -2 & -4 \\ 0 & 8 & -19/2 & -16 \end{matrix}$$

So $\det(A) = 4\det(D)$

(Multiple row ops on D)
Use row operation -3 * Row 2 + Row 3
-8 * Row 2 + Row 4 on D to matrix E.

$\Rightarrow \det(E) = \det(D)$

E =

$$\begin{matrix} 1 & 0 & 1/2 & 2 \\ 0 & 1 & -11/4 & -4 \\ 0 & 0 & 25/4 & 8 \\ 0 & 0 & 25/2 & 16 \end{matrix}$$

So $\det(A) = 4\det(E)$

Use row operation -2 * Row 3 + Row 4 on E to get matrix F. $\Rightarrow \det(F) = \det(E)$

F =

$$\begin{matrix} 1 & 0 & 1/2 & 2 \\ 0 & 1 & -11/4 & -4 \\ 0 & 0 & 25/4 & 8 \\ 0 & 0 & 0 & 0 \end{matrix}$$

So $\det(A) = 4\det(F)$

But $\det(F) = 0$

So $\det(A) = 0$.

Sec. 3.1 Use the method of row operations.

#18(b)

A =

4	1	3
2	3	0
1	3	2

Use row operation Row 1 <--> Row 3 on A to get matrix B. $\Rightarrow \det(B) = -\det(A)$

B =

1	3	2
2	3	0
4	1	3

So $\det(A) = -\det(B)$

Use row operation -2 * Row 1 + Row 2 on B to matrix C. $\Rightarrow \det(C) = \det(B)$

C =

1	3	2
0	-3	-4
4	1	3

So $\det(A) = -\det(C)$

Use row operation -4 * Row 1 + Row 3 on C to get matrix D. $\Rightarrow \det(D) = \det(C)$

D =

1	3	2
0	-3	-4
0	-11	-5

So $\det(A) = -\det(D)$

Use row operation -11/3 * Row 2 + Row 3 to get matrix E. $\Rightarrow \det(E) = \det(D)$

E =

1	3	2
0	-3	-4
0	0	29/3

So $\det(A) = -\det(E)$

Since $\det(E) = -29$

$\det(A) = 29$

Sec. 3.1 Use reduction to upper triangular form.

18 (c)

A =

1	2	3
2	1	0
-3	1	2

Use row operation -2 * Row 1 + Row 2 on A to get matrix B. $\Rightarrow \det(B) = \det(A)$.

B =

1	2	3
0	-3	-6
-3	1	2

$\det(A) = \det(B)$

Use row operation 3 * Row 1 + Row 3 on B to get matrix C. $\Rightarrow \det(C) = \det(B)$

C =

1	2	3
0	-3	-6
0	7	11

$\det(A) = \det(C)$

Use row operation 7/3 * Row 2 + Row 3 on C to get matrix D. $\Rightarrow \det(D) = \det(C)$

D =

1	2	3
0	-3	-6
0	0	-3

$\det(A) = \det(D)$

$= \boxed{9}$

Note we used only row operations that didn't change the value of the determinant.

#22 $\det(A) = -4$

(a) $\det(A^2) = \det(A)\det(A) = 16$

(b) $\det(A^4) = \det(A^2)\det(A^2) = (16)^2 = 256$

(c) $\det(A^{-1}) = -1/4$

#23 $\det(A) = 2 \quad \det(B) = -3$

$$\det(A^{-1}B^T) = \det(A^{-1})\det(B^T)$$

$$= \frac{1}{\det(A)}\det(B) = -3/2$$

Sect 3.2 Use reduction to triangular form

#20 (c)

A =

$$\begin{matrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ 1 & 4 & 3 \end{matrix}$$

Replacement by Linear Combination Complete: -3 * Row 1 + Row 2.
The current matrix is:

B =

$$\begin{matrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 1 & 4 & 3 \end{matrix}$$

$\det(B) = \det(A)$

Replacement by Linear Combination Complete: -1 * Row 1 + Row 3.
The current matrix is:

C =

$$\begin{matrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 2 & 4 \end{matrix}$$

$\det(C) = \det(B) = \det(A)$

$\boxed{\frac{1}{2}R_2 + R_3}$

D =

$$\begin{matrix} 1.0000 & 2.0000 & -1.0000 \\ 0 & -4.0000 & 3.0000 \\ 0 & 0 & 5.5000 \end{matrix}$$

$\det(D) = \det(C) = \det(A)$

upper triangular so

$\det(A) = (1)(-4)(5.5) = -22$

Sec 3.1 T exercises

T.5. Show if $\det(AB)=0$ then $\det(A)=0$ or $\det(B)=0$.

$\det(AB)=\det(A)*\det(B)$ property of products of det

Thus product of numbers can be zero if and only if one of the numbers, either $\det(A)$ or $\det(B)$ is zero.

T.6. Is $\det(AB)=\det(BA)$?

Yes: $\det(AB) = \det(A)\det(B)$ ← these are numbers so we can commute them

$= \det(B)\det(A)$

$= \det(BA)$ ← by prop. of products of det.

T.8. Show that if $AB = I_n$ then $\det(A) \neq 0$ and $\det(B) \neq 0$.

$\det(AB) = \det(I_n) = 1$ since I_n is diagonal with all 1's on the diagonal

prop. of prod. of det

So then $\det(A)\det(B) = 1$ ⇒ product of two numbers is $=1$

so both must be $\neq 0$.

T.9. Show if $A = A^{-1}$, then $\det(A) = \pm 1$.

Since $A = A^{-1}$ ⇒ $\det(A) = \det(A^{-1})$

but $\det(A^{-1}) = \frac{1}{\det(A)}$

So $\det(A) = \frac{1}{\det(A)}$ ⇒ $(\det(A))^2 = 1$ ⇒ $\det(A) = \pm 1$

T.10 A is nonsingular and $A^2 = A$, then show $\det(A)=1$.

Since A is nonsingular $\det(A) \neq 0$.

Since $A^2 = A$ ⇒ $\det(A^2) = \det(A)$ ⇒ $\det(A)\det(A) = \det(A)$

Since $\det(A) \neq 0$, divide both sides by $\det(A)$ ⇒ $\det(A) = 1$.