Section 8.1 Solutions to assigned problems

#1 (a) $Ax_1 = \begin{bmatrix}3 & -1\\ -2 & 2\end{bmatrix}\begin{bmatrix}r\\ 2r\end{bmatrix} = \begin{bmatrix}3r-2r\\ -2r+4r\end{bmatrix} = \begin{bmatrix}r\\ 2r\end{bmatrix} = 1\begin{bmatrix}r\\ 2r\end{bmatrix} = 1\,x_1$

Since $Ax_1 = 1\,x_1$ we have that x_1 is an assoc. eigenvector of $\lambda = 1$.

(b) $Ax_2 = \begin{bmatrix}3 & -1\\ -2 & 2\end{bmatrix}\begin{bmatrix}r\\ -r\end{bmatrix} = \begin{bmatrix}3r+r\\ -2r-2r\end{bmatrix} = \begin{bmatrix}4r\\ -4r\end{bmatrix} = 4\,x_1$

Since $Ax_2 = 4x_2$, we have that x_2 is an assoc. eigenvector of $\lambda = 4$

2 (a) Proceed as in #1. $Ax_1 = \begin{bmatrix}2 & 2 & 3\\ 1 & 2 & 1\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix} = \begin{bmatrix}2-3\\ 1-1\\ 2-1\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix} = -1\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$

$= -1\,x_1$

(b) Proceed as in #1 $Ax_2 = \begin{bmatrix}2 & 2 & 3\\ 1 & 2 & 1\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}-2\\ -3\\ 2\end{bmatrix} = \begin{bmatrix}-4-6+6\\ -2-6+2\\ -4+6+2\end{bmatrix} = \begin{bmatrix}-4\\ -6\\ 4\end{bmatrix}$

$= 2\begin{bmatrix}-2\\ -3\\ 2\end{bmatrix} = 2x_2$

#3 Technique: compute $\det(\lambda I_3 - A)$

$\det\left(\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 2\\ -1 & 3 & 2\end{bmatrix}\right) = \det\begin{bmatrix}\lambda-1 & -2 & -1\\ 0 & \lambda-1 & -2\\ 1 & -3 & \lambda-2\end{bmatrix}\begin{matrix}\lambda-1 & -2\\ 0 & \lambda-1\\ 1 & -3\end{matrix}$

$= (\lambda-1)(\lambda-1)(\lambda-2) + 4 + 0 - (-1)(\lambda-1) - 6(\lambda-1) - 0$

$= (\lambda-1)(\lambda-1)(\lambda-2) + 4 + (\lambda-1) - 6(\lambda-1)$ Expand and collect terms

$= \lambda^3 - 4\lambda^2 + 5\lambda - 2 + 4 + \lambda - 1 - 6\lambda + 6$

$= \lambda^3 - 4\lambda^2 - 7$

#4 Proceed as in #3

$\det(\lambda I_2 - A) = \det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}\right)$

$= \det\begin{bmatrix}\lambda-2 & -1\\ 1 & \lambda-3\end{bmatrix} = (\lambda-2)(\lambda-3) + 1 = \lambda^2 - 5\lambda + 6 + 1$

$= \lambda^2 - 5\lambda + 7$

Section 8.1 (continued)

#5 $\det(\lambda I_3 - A) = \det\left(\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} - \begin{bmatrix}4 & -1 & 3\\ 0 & 2 & 1\\ 0 & 0 & 3\end{bmatrix}\right)$

$= \det\begin{bmatrix}\lambda-4 & 1 & -3\\ 0 & \lambda-2 & -1\\ 0 & 0 & \lambda-3\end{bmatrix} = (\lambda-4)(\lambda-2)(\lambda-3)$

$= \lambda^3 - 9\lambda + 26\lambda - 24$

#11 <u>Technique</u>: Find char poly; find its roots; find corresponding eigenvectors

$\det(\lambda I_2 - A) = \det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\right) = \det\begin{bmatrix}\lambda-1 & 1\\ -2 & \lambda-4\end{bmatrix}$

$= (\lambda-1)(\lambda-4)+2 = \lambda^2 - 5\lambda + 4 + 2 = \lambda^2 - 5\lambda + 6$ (char. poly)

Find roots (the eigenvalues): $\lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$

So $\lambda = 3$ or $\lambda = 2$

$\boxed{\text{Case } \lambda = 3}$ — solve homog. system $(3I_2 - A)x = 0$

the augmented matrix is $\left[\begin{array}{cc|c}2 & 1 & 0\\ -2 & -1 & 0\end{array}\right]$; find RREF

to get $\left[\begin{array}{cc|c}1 & \frac{1}{2} & 0\\ 0 & 0 & 0\end{array}\right] \Rightarrow x_1 - \frac{1}{2}x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2$

Soln is $x = \begin{bmatrix}\frac{1}{2}x_2\\ x_2\end{bmatrix}$; let $x_2 = r \neq 0$, eigenvector is $\begin{bmatrix}\frac{1}{2}r\\ r\end{bmatrix}$

To avoid fractions choose $r = 2 \Rightarrow$ one eigenvector is $\begin{bmatrix}1\\ 2\end{bmatrix}$ corresponding to eigenvalue $\lambda = 3$.

$\boxed{\text{Case } \lambda = 2}$ $(2I_2 - A)x = 0 \Rightarrow \left[\begin{array}{cc|c}1 & 1 & 0\\ -2 & -2 & 0\end{array}\right]$; RREF is

$\left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right] \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$; soln $x = \begin{bmatrix}-x_2\\ x_2\end{bmatrix}$;

Let $x_2 = r \neq 0 \Rightarrow$ one eigenvector is $\begin{bmatrix}r\\ r\end{bmatrix}$; choose $r = 1$.

and $\begin{bmatrix}1\\ 1\end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 2$.

#12 Follow the procedure in #11.

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda-2 & 2 & -3 \\ 0 & \lambda-3 & 2 \\ 0 & 1 & \lambda-2 \end{bmatrix} = (\lambda-2)\det\begin{bmatrix} \lambda-3 & 2 \\ 1 & \lambda-2 \end{bmatrix}$$

(used properties of det here!)

$$= (\lambda-2)\left[(\lambda-3)(\lambda-2)-2\right] = (\lambda-2)(\lambda^2-5\lambda+6-2)$$

$$= (\lambda-2)\underbrace{(\lambda^2-5\lambda+4)}_{\text{factor this}} = (\lambda-2)(\lambda-4)(\lambda-1)$$

So the eigenvalues are $\lambda = 2, 4, 1$.

$\boxed{\text{Case } \lambda=2}$ Solve $(2I_3-A)x=0 \Rightarrow$ augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \text{ RREF is } \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_1 = r \neq 0$ $x = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$; choose $r=1$ then an eigenvector corresponding to $\lambda=2$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{\text{Case } \lambda=4}$ Solve $(4I_3-A)x=0 \Rightarrow$ augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \text{ RREF is } \left[\begin{array}{ccc|c} 1 & 0 & -7/2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} +7/2\, x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

Let $x_3 = r \neq 0$, $x = \begin{bmatrix} 7/2\, r \\ -2r \\ r \end{bmatrix}$; choose $r=2 \Rightarrow x = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$

is an eigenvector corresponding to eigenvalue $\lambda = 4$.

$\boxed{\text{Case } \lambda=1}$ Solve $(1I_3-A)=x \Rightarrow$ augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \text{ RREF is } \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

Let $x_3 = r \neq 0$, $x = \begin{bmatrix} -r \\ r \\ r \end{bmatrix}$; choose $r=1 \Rightarrow x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

is an eigenvector corresponding to eigenvalue $\lambda = 1$

#13 Follow the procedure in #11 & #12.

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-2 & -1 \\ -2 & 2 & \lambda-1 \end{bmatrix} = (\lambda-2)(\lambda-2)(\lambda-1) - 4 + 6 - 6(\lambda-2) + 2(\lambda-2) - 2(\lambda-1)$$

$$= (\lambda-2)(\lambda-2)(\lambda-1) - 4(\lambda-2) - 2(\lambda-1) + 2$$

$$= (\lambda^2 - 4\lambda + 4)(\lambda-1) - 4\lambda + 8 - 2\lambda + 2 + 2$$

$$= \begin{array}{l} \lambda^3 - 4\lambda^2 + 4\lambda \\ \quad - \lambda^2 + 4\lambda - 4 \\ \qquad - 6\lambda + 12 \end{array} = \lambda^3 - 5\lambda^2 + 2\lambda + 8$$

We need to factor this!

"Observe" that if $\lambda = -1$, then $\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$ so $(\lambda+1)$ is a factor of the char. poly.

The other factor is $\lambda^2 - 6\lambda + 8$; thus factors as $(\lambda-4)(\lambda-2)$

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 8 \\ \hline \lambda+1 & \lambda^3 - 5\lambda^2 + 2\lambda + 8 \\ & \lambda^3 + 1\lambda^2 \\ \hline & \quad -6\lambda^2 + 2\lambda \\ & \quad -6\lambda^2 - 6\lambda \\ \hline & \qquad 8\lambda + 8 \end{array}$$

So the char. poly is $(\lambda+1)(\lambda-4)(\lambda-2)$.

We find that the eigenvalues are $\lambda = -1, 4, 2$

By setting up and solving the corresponding homog. systems $(\lambda I_3 - A)x = 0$ we find that corresponding eigenvectors are:

for $\lambda = -1$ $\quad \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = 4$ $\quad \begin{bmatrix} 4 \\ 2.5 \\ 1 \end{bmatrix}$

for $\lambda = 2$ $\quad \begin{bmatrix} -1 \\ -1.5 \\ 1 \end{bmatrix}$

Section 8.1

\# I1 We show S is closed under + and scalar multiplication.

Let x, y be in S $\Rightarrow$ $Ax = \lambda_j x$ and $Ay = \lambda_j y$

Then $A(x+y) = Ax + Ay = \lambda_j x + \lambda_j y = \lambda_j (x+y)$

So x+y is in S.

Let r be any scalar; $A(rx) = rAx = r\lambda_j x = \lambda_j rx$

Then rx is in S; if r=0 since zero vector is in S

it still means we are closed under scalar mult.

\# 21 Solve $(2I_3 - A)x = 0$ $\Rightarrow$ aug matrix is $\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

The RREF is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

thus $x = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$ Let $x_3 = r \neq 0$ then $x = \begin{bmatrix} -r \\ 0 \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So we can take $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ as a basis for the eigenspace corresponding to $\lambda = 2$

\# 22 Proceed as in #21 $(3I_3 - A)x = 0$ has aug. matrix

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \Rightarrow$ RREF is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix}$

Let $x_2 = r$, $x_3 = s$ $\Rightarrow x = \begin{bmatrix} -s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

So $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace for $\lambda = 3$.