Independence/Dependence Notes

Let V be one of the following: R^n, M_{mn}, or P_n. (Recall that each of these is a closed set under their standard operations of addition and scalar multiplication.)

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. (Here we consider only finite subsets of V.)

Definition We say that $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is a **linearly dependent** subset of V provided that there exist scalars $c_1, c_2, \ldots, c_k$ NOT all zero so that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \sum_{j=1}^{k} c_j\mathbf{v_j} = \mathbf{0} \Leftarrow \text{the zero in set V}.$$

If the only way that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \sum_{j=1}^{k} c_j\mathbf{v_j} = \mathbf{0}$ is when $c_1 = c_2 = \cdots = c_k = 0$ we say that S is a **linearly independent** subset of V.

Why is the terminology dependent & independent used?

One explanation: If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ and there are scalars $c_1, c_2, \ldots, c_k$ NOT all zero so that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \sum_{j=1}^{k} c_j\mathbf{v_j} = \mathbf{0}$, then at least one of the c's is not zero. Suppose that $c_i \neq 0$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_{i-1}\mathbf{v_{i-1}} + c_i\mathbf{v_i} + c_{i+1}\mathbf{v_{i+1}} + \cdots + c_k\mathbf{v_k} = \sum_{j=1}^{k} c_j\mathbf{v_j} = \mathbf{0}$.

Rearranging we have $c_i\mathbf{v_i} = -\sum_{\substack{j=1 \\ j\neq i}}^{k} c_j\mathbf{v_j}$ or equivalently that $\mathbf{v_i} = -\frac{1}{c_i}\sum_{\substack{j=1 \\ j\neq i}}^{k} c_j\mathbf{v_j}$. This implies that $\mathbf{v_i}$ is a linear combination of the other vector in set S; that is, $\mathbf{v_i}$ ***'depends'*** on the other vectors.

Example 1. Let V = R^3 and $\mathbf{S} = \{(1,2,-1), (1,-2,1), (-3,2,-1)\}$. Is S a linearly dependent or linearly independent subset of V?

Strategy: We form a linear combination of the vectors, set it equal to zero, and construct a homogeneous linear system. Find the RREF to determine the solution set. If there is a nontrivial solution then S is linearly dependent, otherwise it is linearly independent.

We have the linear combination

$$c_1(1,2,-1)+c_2\ (1,-2,1)+c_3\ (-3,2,-1)=(0,0,0).$$

Performing the scalar multiplications and adding we get

$$(c_1+c_2-3c_3,2c_1-2c_2+2c_3,-c_1+c_2-c_3)=(0,0,0).$$

Equating corresponding entries we have the homogeneous system

$$\begin{aligned} c_1+c_2-3c_3 &= 0 \\ 2c_1-2c_2+2c_3 &= 0 \\ -c_1+c_2-c_3 &= 0 \end{aligned} \quad \text{with augmented matrix} \left[\begin{array}{rrr|r} 1 & 1 & -3 & 0 \\ 2 & -2 & 2 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right].$$

The RREF is $\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ and it follows there is a nontrivial solution so that set is linearly dependent.

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<u>Example 2.</u> Let V = P_2 and $\mathbf{S}=\left\{\mathbf{t}^2+\mathbf{t}+2,2\mathbf{t}^2+\mathbf{t},2\mathbf{t}+1\right\}$. Is S a linearly dependent or linearly independent subset of V?

Following the same strategy as in Example 1 we have

$$c_1\left(\mathbf{t}^2+\mathbf{t}+2\right)+c_2\left(2\mathbf{t}^2+\mathbf{t}\right)+c_3\left(2\mathbf{t}+1\right)=0.$$

Expanding and collecting like terms we have

$$(c_1+2c_2)\mathbf{t}^2+(c_1+c_2+2c_3)\mathbf{t}+(2c_1+c_3)=0.$$

Equating coefficients to zero we get the linear system $\begin{aligned} c_1+2c_2 &= 0 \\ c_1+c_2+2c_3 &= 0 \\ 2c_1+c_3 &= 0 \end{aligned}$ which has augmented matrix $\left[\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$ and its RREF is $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$. It follows there is only the trivial solution so that set S is linearly independent.

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Example 3. The homogeneous linear system $\mathbf{Ax = 0}$, where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}$ has general solution $\mathbf{x} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ so the set $S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans the solution set. Is S a linearly dependent or linearly independent subset of R^4?

Following the same strategy as in Example 1 we have $c_1\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Writing this as a matrix vector product we get linear system $\begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ whose augmented matrix is

$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$. Computing the RREF we get $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ and it follows that the only solution is $c_1 = c_2 = 0$ hence S is a linearly independent set.

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Example 4. Let V = M_{22} and $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \right\}$. Is S a linearly dependent or linearly independent subset of V?

Following the same strategy as in Example 1 we have $c_1\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Performing the scalar multiplication and then adding the matrices we get the expression

$$\begin{bmatrix} c_1+c_3 & 2c_1+c_2+c_3 \\ 0 & c_1-c_2+2c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equating corresponding entries gives the system of equations

$$\begin{aligned} c_1+c_3 &= 0 \\ 2c_1+c_2+c_3 &= 0 \\ c_1-c_2+2c_3 &= 0 \end{aligned}$$

which is equivalent to the matrix equation $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Forming the augmented matrix and computing the RREF we get $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. It follows that there is an arbitrary constant in the general solution so there exist scalars c_1, c_2, and c_3 not all zero so that

$$c_1\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so the set is linearly dependent.

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Results for NONSINGULAR and SINGULAR matrices:

If **A** is n x n and **nonsingular**, then the columns of **A** are linearly independent. Explain why.
Is this also true for the rows of **A**?

If **A** is n x n and **singular,** then the columns of A are linearly dependent. Explain why.
Is this also true for the rows of **A**?

In some **SPECIAL CASES** we can determine if a set is linearly dependent by **inspection**.

- If a set contains the zero vector, then the set is linearly dependent.
- If one vector in a set is a scalar multiple of another vector, then the set is linearly dependent.
- If one vector in a set is a linear combination of other vectors in the set, then the set is linearly dependent. (This may not be easy to see merely by inspection.)

Particular linearly independent sets.

- {**i**, **j**}
- {**i**, **j**, **k**}
- $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_k\}$ $k \le n$ where $\mathbf{e}_1 = \mathbf{col}_1(\mathbf{I}_n)$, $\mathbf{e}_2 = \mathbf{col}_2(\mathbf{I}_n)$, ... , $\mathbf{e}_k = \mathbf{col}_k(\mathbf{I}_n)$
- The columns of any size identity matrix.
- The rows of any size identity matrix.
- The columns (and rows) of any diagonal matrix with all nonzero diagonal entries.

Linear Independence_ Dependence Exercises

1. Determine if $\mathbf{S} = \{(4,2,1),(2,6,-5)(1,-2,3)\}$ is a linearly dependent or linearly independent set of R^3.

2. Determine if $\mathbf{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly dependent or linearly independent set of R^3.

3. Determine if $\mathbf{S} = \{t^2 + 1, \mathbf{t} - 2, \mathbf{t} + 3\}$ is a linearly dependent or linearly independent set of P_2.

4. Determine if $\mathbf{S} = \{3\mathbf{t} + 1,\ 3\mathbf{t}^2 + 1,\ 2\mathbf{t}^2 + \mathbf{t} + 1\}$ is a linearly dependent or linearly independent set of P_2.

5. Determine if $\mathbf{S} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}$ is a linearly dependent or linearly independent set of M_{22}.

6. Suppose that matrix **A** is 4 × 6 provide an argument that the columns of **A** are linearly dependent.

7. Let C be the set of all real continuous functions on $(-\infty, \infty)$ with the usual operations of addition of functions and multiplication by real scalars. Show that $\mathbf{S} = \{\sin^2(t), \cos^2(t), \cos(2t)\}$ is a linearly dependent set of C.

8. Let **A** be a 5 × 5 nonsingular matrix give an argument that its columns are a linearly independent set of R^5.

9. Let **A** be a 5 × 5 singular matrix give an argument that its columns are a linearly dependent set of R^5.

10. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$. Find a set of vectors that span the set of solutions to **Ax = 0** and show that they are linearly independent.