Linear Independence_ Dependence Exercises Solutions

1. Determine if $\mathbf{S}=\{(4,2,1),(2,6,-5)(1,-2,3)\}$ is a linearly dependent or linearly independent set of R^3.
   Form the linear combination $\mathbf{c}_1(4,2,1)+\mathbf{c}_2(2,6,-5)+\mathbf{c}_3(1,-2,3)=(0,0,0)$. Expand, simplify, and equate corresponding entries to get the homogeneous linear system $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 6 & -2 \\ 1 & -5 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; form the augmented matrix and compute its RREF. The RREF is

1	0	1/2	|	0
0	1	-1/2	|	0
0	0	0	|	0

Since there are 3 unknowns and only 2 nonzero equations there are infinitely many solutions, **S is a linearly dependent set of $\mathbf{R^3}$.**
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2. Determine if $\mathbf{S}=\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a linearly dependent or linearly independent set of R^3.

   Form the linear combination $\mathbf{c}_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}+\mathbf{c}_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+\mathbf{c}_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Expand, simplify, and equate corresponding entries to get the homogeneous linear system $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; form the augmented matrix and compute its RREF. The RREF is

1	0	0	|	0
0	1	0	|	0
0	0	1	|	0

So we have only the zero solution, hence **S is a linearly independent set in $\mathbf{R^3}$.**
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3. Determine if $\mathbf{S}=\{\mathbf{t}^2+1,\mathbf{t}-2,\mathbf{t}+3\}$ is a linearly dependent or linearly independent set of P_2.

Note: we can associate a vector from R^3 with each member of S; these vectors are respectively $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$. Then we can form a linear combination of these vectors and quickly get to the linear system shown below. Or we can proceed as shown next.

Form the linear combination $\mathbf{c}_1\left(\mathbf{t}^2+1\right)+\mathbf{c}_2\left(\mathbf{t}-2\right)+\mathbf{c}_3\left(\mathbf{t}+3\right)=0$. Expand, simplify, and equate coefficients of like degree terms to get the homogeneous linear system $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & -2 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; form the augmented matrix and compute its RREF. The RREF is

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

So we have only the zero solution, hence **S is a linearly independent set.**

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4. Determine if $\mathbf{S}=\left\{3\mathbf{t}+1, 3\mathbf{t}^2+1, 2\mathbf{t}^2+\mathbf{t}+1\right\}$ is linearly dependent or linearly independent set in P_2. Associate a vector from R^3 with each member of S; these vectors are, respectively, $\begin{bmatrix}0\\3\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$. Then we can form a linear combination of these vectors and quickly get to the linear system $\begin{bmatrix}0 & 3 & 2\\3 & 0 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; form the augmented matrix and compute its RREF. The RREF is

$$\left[\begin{array}{ccc|c}1 & 0 & 1/3 & 0\\0 & 1 & 2/3 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Since there are 3 unknowns and only 2 nonzero equations there are infinitely many solutions, hence **S is a linearly dependent set.**

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5. Determine if $\mathbf{S}=\left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}\right\}$ is linearly dependent or linearly independent set of M_{22}. With each matrix in S we will associate a vector in R^4 by “stacking the columns’; here we associate the vectors respectively as $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix}$. Now form a linear combination of these vectors and set it equal to zero; the corresponding homogeneous system is $\begin{bmatrix}1 & 1 & 0\\1 & 0 & 0\\1 & 0 & 1\\1 & 2 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$. Form the augmented matrix and compute its RREF; we get

1	0	0	|	0
0	1	0	|	0
0	0	1	|	0
0	0	0	|	0

It follows that $c_1 = c_2 = c_3 = 0$, so **S is a linearly independent set of M_{22}.**

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6. Suppose that matrix **A** is 4 × 6 provide an argument that the columns of **A** are linearly dependent.
   If you form the augmented matrix **[A | 0]**, the corresponding linear system is equivalent to a linear combination of the columns of A. **Since RREF of [A | 0] will have more unknowns than nonzero equations, there will be infinitely many solutions.** Hence certainly there is a linear combination of the columns with nonzero coefficients that gives the zero vector. So the set of columns is linearly dependent.

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7. Let C be the set of all real continuous functions on $(-\infty, \infty)$ with the usual operations of addition of functions and multiplication by real scalars. Show that $\mathbf{S} = \{\sin^2(t), \cos^2(t), \cos(2t)\}$ is a linearly dependent set of C.

There is a trig identity that says that $\cos(2t) = \cos^2(t) - \sin^2(t)$. Rearranging gives a linear combination in which some coefficients are not zero so that we produce the zero function; $\cos(2t) - \cos^2(t) + \sin^2(t) = 0$.

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8. Let **A** be a 5 × 5 nonsingular matrix give an argument that its columns are a linearly independent set of R^5.

Forming a linear combination of the columns of **A** is equivalent to the matrix vector product **Ac**, where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$. Set **Ac = 0** then form the augmented matrix **[A | 0]**. The RREF of **[A|0]** is $[\mathbf{I_5}|\mathbf{0}]$ since **A** is nonsingular. It follows that all the coefficients must be zero. So the columns of nonsingular matrix **A** are linearly independent.

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9. Let **A** be a 5 × 5 singular matrix give an argument that its columns are a linearly dependent set of R^5.

Forming a linear combination of the columns of **A** is equivalent to the matrix vector product **Ac**, where $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$. Set **Ac = 0** then form the augmented matrix **[A | 0]**. The RREF of **[A|0]** has at least one zero row since **A** is singular. It follows that there will infinitely many solutions so some set of coefficients can be chosen so they are not all zero. So the columns of singular matrix **A** are linearly dependent.

10. The **rref** of the augmented matrix **[A | 0]** where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$ is

1	0	0	-1	|	0
0	1	2	0	|	0
0	0	0	0	|	0
0	0	0	0	|	0

So $x_1 - x_4 = 0$, $x_2 + 2x_3 = 0$ hence we have solution vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ -2x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So S = $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a spanning set for the set of solutions to **[A | 0]**. Setting $x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, a little algebra shows that $x_3 = x_4 = 0$, so set S is linearly independent.