Review Problems for Final in Linear Algebra Fall 2009 Solutions

1. Let $u = \begin{bmatrix} 1 & 2 & 3 & 0 \end{bmatrix}$ and $v = \begin{bmatrix} 2 & -1 & 0 & 1 \end{bmatrix}$.

(a) Find a nonzero vector w that is orthogonal to both u and v.

Set up a homogeneous linear system using a coefficient matrix with rows u and v. Find its RREF and write the general solution. You should get $r\begin{bmatrix} -0.6 \\ -1.2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -0.4 \\ 0.2 \\ 0 \\ 1 \end{bmatrix}$ where r and s are arbitrary real numbers. For example choose r = s = 1 and then one vector w that works is $w = \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}$.

(b) How many such vectors w are there? Explain.

Infinitely many. You can't choose r = s = 0.

2. Let **A** be a 6 × 6 matrix with row 6 equal to the sum of the first 5 rows. For each of the following topics discuss properties of matrix **A**. {det, rank, rref, singular/nonsingular, eigenvalues, diagonalizable, null space, row space, defective}

det(**A**) = 0, rank(**A**) < 6, rref of **A** has at least one zero row, **A** is singular, zero is an eigenvalue, unable to tell if it is diagonalizable, null space has infinitely many vectors, dimension of the row space < 6, unable to tell if it is defective

3. Let $\mathbf{A} = \begin{bmatrix} a & 0 & b & c \\ 0 & k & 0 & 0 \\ d & 0 & e & f \\ g & 0 & h & m \end{bmatrix}$. Show that k is an eigenvalue of **A**.

We look for an eigenvector x so that Ax = kx. Let $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Then Ax = kx, so k is an eigenvalue.

4. System Ax = b has

$$\text{rref}([A \mid b]) = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 4 & 5 \\ 0 & 0 & 1 & 7 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

. Find a basis for the null space of **A**.

Just find x_h, the solution of the homogeneous system, and express the solution as a linear combination of vectors with coefficients which are arbitrary constants. The vectors involved will give the basis. Vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -7 \\ 1 \end{bmatrix}$ are a basis for the null space.

5. Let $\mathbf{A} = \begin{bmatrix} 1 & x & 1 \\ 0 & -1 & x \\ 2 & 2 & 1 \end{bmatrix}$. (a) Determine all values of x so that det(**A**) = 1.

det(A) = $2x^2 - 2x + 1$; it equals 1 for x's that satisfy $2x^2 - 2x = 0$; so x = 0 or x = 1.

(b) Find the value of x where det(**A**) is a minimum.

Compute $\frac{d}{dx}\left(2x^2 - 2x + 1\right) = 4x - 2$. Set it equal to zero and solve for x; we get x = ½. Note that the second derivative at this critical point is 4 which is positive so we have a minimum at x = ½. Coincidentally det(A) = ½ when x = ½.

(c) For x a real number, can **A** ever be a singular matrix?

NO, the smallest value of the determinat is ½ so it can never be zero.

6. Explain how to construct a matrix **A** with nonzero entries so that its eigenvalues are 0, 1, 2.

Find a nonsingular matrix P so that P^{-1} times a diagonal matrix with diagonal entries 0, 1, 2 times P contains no zero entries.

7. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$ and vectors $u = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$. Show that u and v are eigenvectors of **A** and find the corresponding eigenvalues.

Just compute **Au** and note you get 2u so 2 is an eigenvalue. Similarly compute Av and note you get 3v so 3 is an eigenvalue.

8. Which of the following sets span R^3? (a) $S = \{\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 & -1 \end{bmatrix}\}$.

(b) $T = \{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 6 & 4 & 6 \end{bmatrix}\}$.

For part (a) form a matrix with the vectors as rows and adjoin a column with letter say a, b, c. Compute its RREF. You will find that system is inconsistent for certain choices of a, b, c. Thus NO. For part (b) use a similar approach. This time you will find that the system is consistent for all choices of a, b, c. Thus YES.

9. Determine which of the following are subspaces.

(a) S = all continuous functions f(x) so that f(5) = 6. **NO, it is not closed.**

(b) W = all 3 × 3 matrices **A** such that $\mathbf{A}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **NO, it is not closed.**

(c) W = all 2 × 3 matrices **A** whose null space includes the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. **YES.**

(d) W = all 3 × 1 matrices x so that the dot product of x and $\begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$ is zero. **YES.**

10. Let W be the subspace of 3 × 3 matrices with 2^{nd} row and 2^{nd} column all zero. Find a basis for W.

Find a basis for W. One basis is $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$