

SPAN NOTES

Definition A **linear combination** of two or more quantities is a **sum of scalar multiples** of the quantities. The linear combination in which each scalar is zero is called the **trivial linear combination**.

Example 1.

a) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Then $2\mathbf{b} - \mathbf{c} + 3\mathbf{d}$ is a linear combination of the 2-vectors \mathbf{b} , \mathbf{c} , and \mathbf{d} with coefficients 2, -1, and 3 respectively. We have

$$2\mathbf{b} - \mathbf{c} + 3\mathbf{d} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 1 \end{bmatrix}.$$

b) Let $\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{A}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3 + c_4\mathbf{A}_4$ is a linear combination of the 2×2 matrices \mathbf{A}_i , $i = 1, \dots, 4$ with coefficients c_1 , c_2 , c_3 , and c_4 respectively.

c) Polynomial $2x^3 - 7x^2 + 8x + 4$ is a linear combination of functions x^3 , x^2 , x , and 1 with coefficients 2, -7, 8, and 4 respectively. ■

Span

Linear combinations play an important role in our analysis of the information contained in a matrix that represents a linear system of equations or other processes. Here we discuss two important ideas related to linear combinations.

Definition: The set of all possible linear combinations of a particular set S of quantities is called the **span** of set S and is denoted **span(S)**.

If it is possible to find scalars so that a given quantity can be expressed as a linear combination of a particular set S , we say that the given quantity is **in the span** of the set S .

For a given set S , we often want to give a concise description of the **span(S)**.

These ideas use the word span as **noun**; the full extent or stretch of the set S .

The kind of sets S we will use include \mathbf{R}^n , $\mathbf{m} \times \mathbf{n}$ matrices \mathbf{M}_{mn} , and **functions** or subsets of these.

Example 2. Determine if \mathbf{v} is in $\text{span}(\mathbf{S})$ for each of the following.

$$(a) \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}. \quad (b) \mathbf{v} = \begin{bmatrix} -1 & 4 \\ 4 & -4 \end{bmatrix}, \mathbf{S} = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}.$$

$$(c) \mathbf{v} = t^2 - t + 3, \mathbf{S} = \{t + 1, t^2 - 2, t^2 + 2t\}.$$

■

Example 3. Give a concise description of the $\text{span}(\mathbf{S})$ for each of the following sets \mathbf{S} .

$$(a) \mathbf{S} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}. \quad (b) \mathbf{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \quad (c) \mathbf{S} = \{t, 1\}.$$

$$(d) \mathbf{S} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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Here is another issue regarding span.

Let W be a set whose members are described for you and for which addition and scalar multiplication are defined. (Often W will contain infinitely many members.) Let \mathbf{S} be a subset of W . (Often \mathbf{S} contains just a few members.)

We say **$\text{span}(\mathbf{S}) = W$** provided every member of W is a linear combination of the members of \mathbf{S} . In this case we say **\mathbf{S} spans W** or **\mathbf{S} is a spanning set of W** .

This idea uses the word span as **verb**; to extend or reach across.

Example 4. (a) Let $W = \mathbb{R}^2$. Then $\mathbf{S} = \{i, j\}$ is a spanning set of W ; $\text{span}(\mathbf{S}) = \mathbb{R}^2$.

(b) Let $W = P_2$, the set of all polynomials of degree two or less. Then $\mathbf{S} = \{t^2, t, 1\}$ is a spanning set of P_2 ; $\text{span}\{t^2, t, 1\} = P_2$.

(c) Let W = the set of all 2×2 matrices which is denoted by M_{22} . Then

$$\mathbf{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } W.$$

■

In Example 4 it was easy to see that the set S spanned W. In fact the sets S in Example 4 are referred to as the **natural spanning set** of the corresponding set W. Next in Examples 5 - 7 we must check things very carefully to determine if S is a spanning set of W or not..

Our **strategy** is form a linear combination of the members of set S using **unknown** coefficients and then set it equal to an expression that represents an arbitrary member of W. From this we will derive a linear system of equations to determine the unknown coefficients of the linear combination. **If the linear system is always consistent then the set S spans W. If the linear system is inconsistent for some choices of members in W, then the set S does not span W.**

Example 5. Let $W = \mathbb{R}^3$ and $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \right\}$. Is $\text{span}(S) = \mathbb{R}^3$? Here we form the linear

combination of the members of S with unknown coefficients $c_1, c_2,$ and c_3 and set it equal to an arbitrary member of \mathbb{R}^3 which we represent by $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where a, b, and c can be any real numbers.

We set the linear combination of the members of S equal to an arbitrary member of \mathbb{R}^3 and we

get $c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. This expression is equivalent to the linear system

$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Next form the augmented matrix and compute its RREF. We find that the

RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & -2a+b \\ 0 & 0 & 0 & a-b+c \end{array} \right]$$

The linear system is inconsistent if $a - b + c$ is not zero. Since a, b, and c can be chosen arbitrarily, there are members of $W = \mathbb{R}^3$, which can not be written as linear combination of the members of S. Thus S does not span \mathbb{R}^3 . ■

Example 6. Let $W = M_{22}$, the set of all 2×2 matrices, and $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Is $\text{span}(S) = M_{22}$? Here we form the linear combination of the members of S with unknown

coefficients c_1, c_2, c_3 and c_4 and set it equal to an arbitrary member of M_{22} which we represent by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c,$ and d can be any real numbers. We obtain

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a \\ 0 & 0 & 2 & 0 & c \\ 1 & 1 & 0 & 0 & b \\ 1 & 1 & 1 & 1 & d \end{array} \right]$. Computing

its RREF we get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a + \frac{c}{2} + b - d \\ 0 & 1 & 0 & 0 & -a - \frac{c}{2} + d \\ 0 & 0 & 0 & 1 & \frac{c}{2} \\ 0 & 0 & 0 & 0 & \frac{-c}{2} - b + d \end{array} \right]$. Since the

corresponding system is consistent for any member of M_{22} , we have that $\text{span}(\mathbf{S}) = M_{22}$. ■

Example 7. Let $W = P_2$, the set of all polynomials of degree two or less, and

$S = \{t^2 + t, t + 2, t^2 - 1\}$. Is $\text{span}(S) = P_2$? We form the linear combination of the members of S with unknown coefficients $c_1, c_2,$ and c_3 and set it equal to an arbitrary member of P_2 which we represent by $at^2 + bt + c$. We have

$$c_1(t^2 + t) + c_2(t + 2) + c_3(t^2 - 1) = at^2 + bt + c.$$

Expanding and collecting terms containing like powers of t on the left side we have

$$(c_1 + c_3)t^2 + (c_1 + c_2)t + (2c_2 - c_3) = at^2 + bt + c.$$

Since two polynomials are equal if and only if the coefficients of corresponding terms are equal, we equate coefficients of like powered terms to get the following system of equations

$$c_1 + c_3 = a$$

$$c_1 + c_2 = b$$

$$2c_2 - c_3 = c$$

Expressing this system in matrix form and displaying the augmented matrix we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 2 & -1 & c \end{array} \right].$$

Computing the REEF we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -a+2b-c \\ 0 & 1 & 0 & a-b+c \\ 0 & 0 & 1 & 2a-2b+c \end{array} \right]$$

and it follows that the linear system is consistent for any choice of a, b, and c, hence every member of P_2 can be expressed as a linear combination of the vectors of S, so $\text{span}(\mathbf{S}) = P_2$. ■

Example 8 gives another type of spanning exercise.

Example 8. Find a spanning set for the set of all solutions to the $\mathbf{Ax} = \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}.$$

Strategy: find the RREF of $[\mathbf{A} \mid \mathbf{0}]$ and express the solution as a linear combination of vectors of constants with coefficients arbitrary constants. The vectors of constants will be a spanning set for the set of all solutions.

The RREF of the augmented matrix is $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$. Solving this system we have

$x_1 = -x_2 - 2x_4$
 $x_3 = x_4$ which indicates that x_2 and x_4 can be chosen arbitrarily. Letting $x_2 = r$ and $x_4 = s$

we have that the solution of the homogeneous system is $\begin{matrix} x_1 = -r - 2s \\ x_2 = r \\ x_3 = s \\ x_4 = s \end{matrix}$. It follows that any solution

\mathbf{x} of the system can be expressed as $\mathbf{x} = r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and so $\mathbf{S} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans the

solution set. ■

Exercises All work must be shown.

In Exercises 1-3, determine if \mathbf{v} is in $\text{span}(S)$.

$$1. \mathbf{v} = \begin{bmatrix} 9 \\ 10 \\ -1 \end{bmatrix}, S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

$$2. \mathbf{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}.$$

$$3. \mathbf{v} = 6t^2 + 9t - 3, S = \{t + 1, t^2 - 2, t^2 + 2t\}.$$

In Exercises 4-7, give a concise description of the $\text{span}(S)$.

$$4. S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \quad 5. S = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}. \quad 6. S = \{t^3, t^2, t, 1\}. \quad 7. S = \{i, j, k\}.$$

In Exercises 8-10, determine if $\text{span}(S) = W$.

$$8. S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, W = \mathbf{R}^3. \quad 9. S = \left\{ \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}, W = M_{22}.$$

$$10. S = \{t + 1, t + 2, t^2\}, W = P_2.$$

In Exercises 11 and 12, find a spanning set for the set of all solutions to the homogeneous systems.

$$11. \mathbf{Ax} = \mathbf{0}, \text{ where } \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

$$12. \mathbf{Ax} = \mathbf{0} \text{ where } \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$