Vector Spaces

Mathematics has been called the science of patterns. The identification of patterns and common features in seemingly diverse situations provides us with opportunities to unify information. This approach can lead to the development of classification schemes and structures, which can be studied independently of a particular setting or application. Thus we can develop ideas that apply to each and every member of the class based upon properties that they have in common. The result is called an abstract model or abstract structure. In many ways this can help us work with more difficult and comprehensive ideas.

In previous chapters we focused on a matrices & vectors. We saw that their structure, the arrangement of rows and columns of information, was applicable to a variety of situations.

- We showed how to manipulate matrices to obtain information about the solution set of a linear system.
- Of particular interest was the RREF of a matrix.
- We showed how to write linear transformations from R^n to R^m in terms of matrix transformations.
- We showed how to use matrices to determine the span of a set.
- We showed how to use matrices to determine if a set is linearly independent or linearly dependent.
- We showed how to use members of R^n (n by 1 matrices) to represent polynomials and members of M_{mn}.

Here we study the structure of the familiar sets R^n, M_{mn}, and P_n and others by observing that they share a **common STRUCTURE** **,** even though their individual members are different. This lets us develop an abstract model which focuses on properties that they have in common.

The abstract model we identify is called a **VECTOR SPACE**, regardless of the type of members in the set of object that satisfy its definition.

DEFINITION

A **real vector space** is a set of elements V together with two operations $\oplus$ and $\odot$ satisfying the following properties:

(α) If $\mathbf{u}$ and $\mathbf{v}$ are any elements of V, then $\mathbf{u} \oplus \mathbf{v}$ is in V (i.e., V is closed under the operation $\oplus$).

(a) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$, for $\mathbf{u}$ and $\mathbf{v}$ in V.

(b) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$, for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V.

(c) There is an element $\mathbf{0}$ in V such that

$$\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}, \quad \text{for all } \mathbf{u} \text{ in } V.$$

(d) For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that

$$\mathbf{u} \oplus -\mathbf{u} = \mathbf{0}.$$

(β) If $\mathbf{u}$ is any element of V and c is any real number, then $c \odot \mathbf{u}$ is in V (i.e., V is closed under the operation $\odot$).

(e) $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$, for all real numbers c and all $\mathbf{u}$ and $\mathbf{v}$ in V.

(f) $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$, for all real numbers c and d, and all $\mathbf{u}$ in V.

(g) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$, for all real numbers c and d and all $\mathbf{u}$ in V.

(h) $1 \odot \mathbf{u} = \mathbf{u}$, for all $\mathbf{u}$ in V.

The elements of V are called **vectors**; the real numbers are called **scalars**. The operation $\oplus$ is called **vector addition**; the operation $\odot$ is called **scalar multiplication**.

WARNING: the definitions of addition of vectors and scalar multiplication of vectors can be made in various ways.

Things we know about vector spaces:

- They are **closed** under the two operations.
- If V with two operations $\oplus$ and $\odot$ is vector space, then we can form linear combinations.
- If V with two operations $\oplus$ and $\odot$ is vector space, then we can ask questions about span. For instance, can you find a spanning set?
- If V with two operations $\oplus$ and $\odot$ is vector space, then we can determine if certain sets of members of V are linearly independent or linearly dependent.

Examples of Vector Spaces

Notation for Vector Spaces Appearing in This Section	
R^n,	the vector space of all n-vectors with real components
M_{mn},	the vector space of all $m \times n$ matrices
$F[a, b]$,	the vector space of all real-valued functions that are defined on the interval $[a, b]$
$F(-\infty, \infty)$,	the vector space of all real-valued functions defined for all real numbers
P_n,	the vector space of all polynomials of degree $\leq n$ together with the zero polynomial
P,	the vector space of all polynomials together with the zero polynomial

We are familiar with the definitions of addition of vectors and scalar multiplication of "vectors" for these vector spaces.

We frequently refer to a real vector space simply as a **vector space**. We also write $\mathbf{u} \oplus \mathbf{v}$ simply as $\mathbf{u} + \mathbf{v}$ and $c \odot \mathbf{u}$ simply as $c\mathbf{u}$, being careful to keep the particular operation in mind.

Other Properties satisfied by *every* vector space.

If V is a vector space, then:

(a) $0\mathbf{u} = \mathbf{0}$, *for every* $\mathbf{u}$ *in* V.

(b) $c\mathbf{0} = \mathbf{0}$, *for every scalar* c.

(c) *If* $c\mathbf{u} = \mathbf{0}$, *then* $c = 0$ *or* $\mathbf{u} = \mathbf{0}$.

(d) $(-1)\mathbf{u} = -\mathbf{u}$, *for every* $\mathbf{u}$ *in* V.

NON-STANDARD EXAMPLE of a VECTOR SPACE

Let V be the set of all 2 $\times$1 real matrices with both entries positive. For vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ in V, we define vector addition as

$$\mathbf{v} \oplus \mathbf{w} = \begin{bmatrix} v_1 w_1 \\ v_2 w_2 \end{bmatrix}$$

and scalar multiplication by a real scalar k is defined as

$$k \odot \mathbf{v} = \begin{bmatrix} {v_1}^k \\ {v_2}^k \end{bmatrix}.$$

We claim (V, $\oplus$, $\odot$) is a real vector space. (Its tricky to verify the 10 properties of the definition of a vector space.)

We focus on subsets W of the vector spaces R^n, M_{mn}, P_n, F[a, b], $F(-\infty,\infty)$ that are vector spaces in their own right.

DEFINITION

Let V be a vector space and W a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is called a SUBSPACE of V.

How do you check that a subset W of a vector space V is a subspace?

<u>Answer:</u> Verify that V is closed under addition of vectors and scalar multiplication of vectors using the same definitions as used in V.

SUBSPACES

Every vector space V has at least two subspaces.

- W = the set containing zero vector alone = {0}. (called the zero subspace)
- W = V, the whole vector space.

There is a very easy way to form subspaces of a vector space V.

> Let S be a nonempty subset of V.
> Then W = span(S) will be a subspace of V.

Examples:

- Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, then span(S) is a subspace of R^3. Give a formula for every entry of the this subspace.
- Let $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, then span(S) is a subspace of M_{22}. Give a formula for every entry of the this subspace.

Some examples of subspaces:

- Let V = R^3. Then W = the set of all vectors of the form $\begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$ where a and b are any real numbers is a subspace of V.
- Let V = M_{22}. Then W = all 2 by 2 matrices of the form $\begin{bmatrix} a & a \\ b & 0 \end{bmatrix}$ where a and b are any real numbers is a subspace of V.
- Let V =P_4, all polynomials of degree 4 or less. Then W = P_2 is a subspace of V.

subspaces associated with matrices:

- **Let A be an m by n matrix. Then the span of the rows of matrix A is called the row space of A, denoted row(A), is a subspace of R^n.**

- **Let A be an m by n matrix. Then the span of the columns of matrix A is called the column space of A, denoted col(A), is a subspace of R^m.**

- **Let A be an m by n matrix. Then the set of all solutions of the homogeneous linear system Ax = 0 is a subspace of R^n, call the null space of A; sometimes the notation ns(A) is used.**

Example: **Let** $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

row(A) = span $\{(1\ \ 1\ \ 0\ \ 0), (0\ 1\ 1\ 0)\}$

= all vectors in R^4 of the form

$$\mathbf{a}(1\ 1\ 0\ \ 0) + \mathbf{b}(0\ 1\ 1\ 0) = (\mathbf{a}\ \ \mathbf{a}+\mathbf{b}\ \ \mathbf{b}\ \ 0)$$

col(A) = span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ **= all vectors in R^3 of the form**

$$\mathbf{a}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{b}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{c}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a}+\mathbf{b} \\ \mathbf{b}+\mathbf{c} \\ 0 \end{bmatrix}$$

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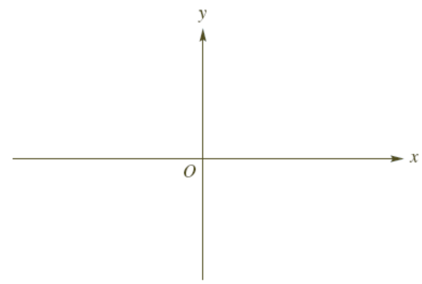
Determine which of the following sets W are subspaces of the vector space V.

- **V = R^2, which geometrically is the Cartesian plane.**

  W = first quadrant; x ≥ 0, y ≥ 0



- **V = M_{22}. W = all matrices of the form $\begin{bmatrix} 0 & 1 \\ x & y \end{bmatrix}$ where x and y are any real numbers.**
- **V = M_{55}. W = all 5 by 5 matrices with at least one zero row.**
- **V = P_2. W = all quadratics whose graph goes through the origin.**
- **V = R^3, which geometrically is 3-space. W = any plane in 3-space.**

General questions:

- **Every vector space must contain what "special" vector?**