The Magnification Factor

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. To find the image of the rectangle R shown in Figure 1 by the matrix transformation determined by **A.**

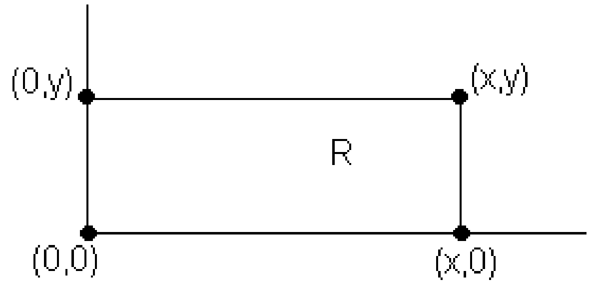

Figure 1.

Let **S** be the matrix whose columns are the vertices of the rectangle

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & x & x & 0 \\ 0 & y & y & 0 & 0 \end{bmatrix}.$$

Then the image of the matrix transformation is

$$f(\mathbf{S}) = \mathbf{AS} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 & x & x & 0 \\ 0 & y & y & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & by & ax+by & ax & 0 \\ 0 & dy & cx+dy & cx & 0 \end{bmatrix}.$$

The vertices in f(**S**) define the parallelogram which is shown in Figure 2.

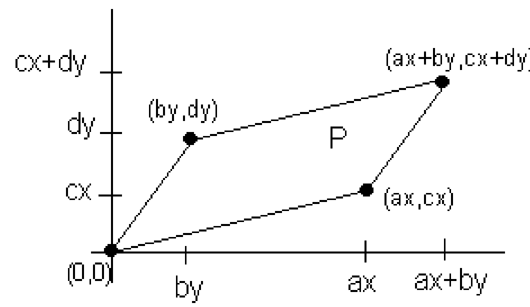

Figure 2.

Next we determine the relationship between the areas of the rectangle R in Figure 1 and the parallelogram P in Figure 2. From Figure 3 we see that

i) The area of the two rectangles labeled **I** is 2|(cx)(by)|.

ii) The area of the two triangles labeled **II** is |(by)(dy)|.

iii) The area of the two triangles labeled **III** is |(ax)(cx)|.

iv) The area of the rectangle enclosing P and the regions labeled **I**, **II**, and **III** is |(ax + by)(cx + dy)|.

Thus

area(P) = |(ax + by)(cx + dy)| - 2|(cx)(by)| - |(by)(dy)| - |(ax)(cx)|

= |ad - bc| |xy|

so the area of the image P is |ad - bc|∗area(R).

It follows that

$$\frac{\text{area(P)}}{\text{area(R)}} = \frac{|ad-bc|xy}{xy} = |ad-bc| \quad . \qquad (1)$$

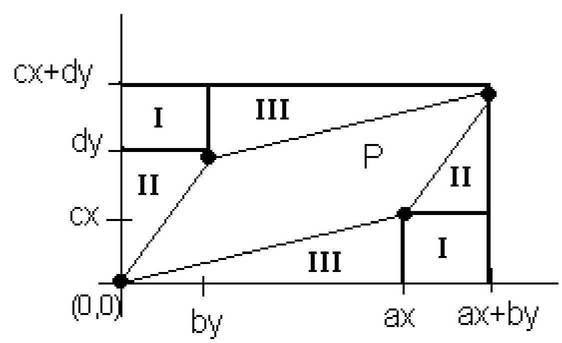

Figure 3.

From (1) we see that the factor by which the area of the image changes depends only on the entries of the matrix **A.** We call this the **magnification factor** of the matrix transformation determined by **A.**