An Oversimplified View of Applied Mathematics

Start ➔ **Real-World Problem, Real-World Phenomena**

Apply axioms, techniques, & information from the underlying discipline to **develop a set of assumptions and set of equations**. This is known as a mathematical model, which will be used for subsequent analysis.

Mathematical Model

Solve the equations; ideally an **analytic** solution is desired, but in most cases an **approximate** solution is the best we can do.

Solution

If the predictions of the model are in agreement with experimental data, the model is accepted and can be used to make predictions regarding situations for which experimental data is not available.

Interpretation of Results

Revise the model if necessary

The objective of this course and our text is to develop methods for determining approximate solution for several classes of mathematical problems that commonly arise during the modeling of real-world phenomena. These include locating roots of a function, determining the value of a definite integral, finding the solution of initial value problems and two-point boundary value problems.

When dealing with approximation methods, there is essential separation into what could be referred to as the **engineering side** of the matter and the **mathematical side**. Issues that arise include:

- Which methods can be applied to which problems?
- What is the best way to implement a particular method?
- Theoretical foundations of how the methods work.
- How well the methods work?
- Under what circumstances the methods can be expected to work?

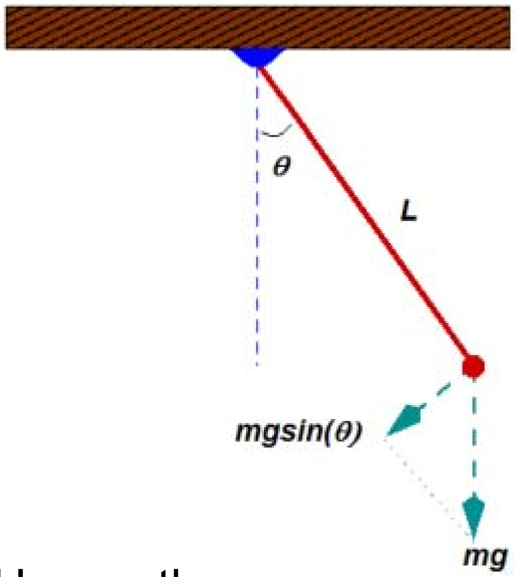
At times the “line” between the two sides is blurry! We consider both sides at times.

Modeling a Simple Pendulum

The pendulum is the classical example of a deterministic system. Its behavior is governed by Newton's law $F = m\,a$, which is a second-order differential equation, since the acceleration a is the second derivative of the displacement.

Assumptions:

The pendulum consists of a point-mass m on a stiff rod of length L which is assumed to have negligible mass.



The pendulum oscillates in a plane, and that the pendulum will undergo small amplitude oscillations and that the energy losses due to air resistance will be negligible.

A little bit of physics gives us: $\mathbf{F = -mg\,sin(\theta) = m\,a}$

The point-mass is at arclength distance θL from the vertical. Hence, the acceleration a is given by the second derivative of θL with respect to time t. Thus we have the equation

$$\theta'' = -g / L \sin(\theta)$$

Let $\omega = \sqrt{g/L}$ and assuming θ is small then $\sin(\theta) \approx \theta$ so the model equation is

$$\theta'' = -\omega^2\,\theta$$

Equation $\theta'' = -\omega^2 \theta$

is a **homogeneous second order linear D.E. with constant coefficients** so it has an analytic solution.

If it turns out that the amplitude of oscillations is not small and that air resistance cannot be neglected, a more appropriate model is given by the equation

$$\theta'' + b\theta' + \omega^2 \sin(\theta) = 0$$

where b is the drag coefficient. The D.E. is **nonlinear** and an exact analytic solution is no longer possible. Approximation techniques are needed.

Crime Scene Investigation

A body is discovered in ab office. At 8PM the coroner determines that the temperature of the corpse is **90°F**. An hour later the temperature of the body is **85°F**. Maintenance reported that the building's air conditioning broke down at 4PM. The computerized climate control system recorded that the office temperature rose at the rate of **1°F** per hour after the malfunction.

Police need to determine the time of death before questioning suspects.

So we will assume that the temperature of the body at time of death was **98.6°F** and began decreasing after death.

We further assume that the cooling follows **Newton's Law of Cooling** which says that the temperature of the body will change at a rate proportional to the difference between the temperature of the body and that of its surroundings,

Let **T(t)** denote the temperature **T** of the body at time *t* measured in hours. Take t = 0 to correspond to 8PM. Thus we know **T(0) = 90** and **T(1) = 85**.

Talking to maintenance we find that the temperature of the office is **72 + t**.

So from Newton's Law of cooling we have the equation

$$\frac{dT}{dt} = -k(T - 72 - t)$$

where k is a positive constant of proportionality.

Equation $$\frac{dT}{dt} = -k(T - 72 - t)$$

can be solved exactly to give $$T(t) = \left(72 + t - \frac{1}{k}\right) + ce^{-kt}$$

where c is a constant of integration. Using **T(0) = 90** we find that $c = 18 + \frac{1}{k}$

Thus the equation becomes

$$T(t) = \left(72 + t - \frac{1}{k}\right) + \left(18 + \frac{1}{k}\right)e^{-kt}$$

Now using that **T(1) = 85** we have an equation involving only **k** given by

$$73 - \frac{1}{k} + \left(18 + \frac{1}{k}\right)e^{-k} = 85$$

This equation cannot be solved for **k**, so we will have to settle for an approximate solution. Once we have a value for **k**, the time of death, t_d, can be approximated by solving

$$98.6 = \left(72 + t_d - \frac{1}{k}\right) + \left(18 + \frac{1}{k}\right)e^{-kt_d}$$

So we need to use a root finding algorithm "twice".