Section 1.2 Exercises 13-15

#13.

Suppose that the sequence $\{p_n\}$ converges linearly to the limit p with asymptotic error constant λ. Further suppose that $p_{n+1} - p$, $p_n - p$ and $p_{n-1} - p$ are all of the same sign. Show that

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} \approx \lambda.$$

Suppose the sequence $\{p_n\}$ converges linearly to p with asymptotic error constant λ. Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda,$$

so, for sufficiently large n, $|p_{n+1} - p| \approx \lambda |p_n - p|$.

Moreover, $|p_n - p| \approx \lambda |p_{n-1} - p|$ or $|p_{n-1} - p| \approx \frac{1}{\lambda}|p_n - p|$.

Because we are given that $p_{n+1} - p$, $p_n - p$ and $p_{n-1} - p$ are all of the same sign, we may drop the absolute values from the above expressions. Now,

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} = \frac{p_{n+1} - p - (p_n - p)}{p_n - p - (p_{n-1} - p)}$$ Explain this step.

$$\approx \frac{\lambda(p_n - p) - (p_n - p)}{p_n - p - \frac{1}{\lambda}(p_n - p)}$$ Explain this step. Where did the lambdas come from?

$$= \frac{\lambda - 1}{1 - \frac{1}{\lambda}} = \lambda.$$ Explain this step.

#14.

A sequence $\{p_n\}$ converges *superlinearly* to p provided $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$

Take this as a definition for superlinear convergence.

Show that if $p_n \to p$ of order α for $\alpha > 1$, then $\{p_n\}$ converges superlinearly to p.

Suppose the sequence $\{p_n\}$ converges p of order $\alpha > 1$ with asymptotic error constant λ. Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda.$$ Give a reason this is true.

Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{|p_{n+1} - p| \cdot |p_n - p|^{\alpha-1}}{|p_n - p|^\alpha}$$ <== Explain how this expression was derived.

$$= \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot \lim_{n\to\infty} |p_n - p|^{\alpha-1}$$

$$= \lambda \cdot 0 = 0.$$

How did lambda get in here?
Where did the zero come from?

Therefore, $\{p_n\}$ converges superlinearly to p.

#15.

Suppose that $\{p_n\}$ converges superlinearly to p (see Exercise 14). Show that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.$$

Note that $\dfrac{p_{n+1} - p_n}{p_n - p} = \dfrac{p_{n+1} - p - (p_n - p)}{p_n - p}$ <== Explain what was done to get this expression.

$= \dfrac{p_{n+1} - p}{p_n - p} - 1.$ <== How did we get this expression?

Because $\{p_n\}$ converges superlinearly to p, it then follows that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = \left| \lim_{n\to\infty} \left(\frac{p_{n+1} - p}{p_n - p} - 1 \right) \right| = |0 - 1| = 1.$$

Explain where Exercise 14 was used.