<u>First Order Linear Differential Equations</u>

A first order linear differential equation (DE) has the following form:

$$\frac{dx(t)}{dt} + a(t)x(t) = b(t)$$

where **a(t)** and **b(t)** are continuous functions of independent variable **t**. We have emphasized the dependence of unknown function **x** by writing **x(t)**. Often the DE is written as $x' + a(t)x = b(t)$. Usually accompany the DE is an initial condition (IC) $x(t_0) = x_0$ and the pair of the DE and IC is referred to as an initial value problem. The IC implies that we want a solution of the DE that goes through the point (t_0, x_0).

It is possible to develop a formula for the solution of the DE as follows. The function

$$e^{\int a(t)dt} \tag{1}$$

is called an **integrating factor**. We multiply both sides of the DE by the integrating factor to obtain

$$e^{\int a(t)dt}\left[\frac{dx(t)}{dt} + a(t)x(t)\right] = e^{\int a(t)dt}b(t)$$

A bit of calculus shows the left side of this expression can be written as the derivative of a certain product. We have

$$\frac{d}{dt}\left(e^{\int a(t)dt}x(t)\right) = e^{\int a(t)dt}b(t)$$

Now we can integrate both sides to obtain

$$e^{\int a(t)dt}x(t) = \int e^{\int a(t)dt}b(t)\ dt + C$$

Then solving for **x(t)** we have

$$x(t) = \frac{\int e^{\int a(t)dt}b(t)\ dt + C}{e^{\int a(t)dt}} \tag{2}$$

As you can see we need to be able to compute two integrals in order to use this expression for the solution of the DE. Assuming we can perform these integrations, we then use the IC $x(t_0) = x_0$ to determine C, the constant of integration.

Once we recognize the DE as first order linear and rearrange it into the form $\frac{dx(t)}{dt} + a(t)x(t) = b(t)$ to identify **a(t)** and **b(t)** we can proceed by determining the integrating factor from Equation (1), then compute the integral $\int e^{\int a(t)dt}b(t)\ dt$ and finally use the expression in Equation (2). Naturally if either of the integrals cannot be computed then Equation (2) is will not provided a “closed form” solution for the DE.

<u>Example:</u> Page 41 #17

Consider the linear, first-order differential equation

$$\frac{dx}{dt} - \frac{1}{t}x = t\sin t.$$

(a) Solve this equation subject to the initial condition $x(\pi/2) = x_0$.

(b) Solve this equation subject to the perturbed initial condition $x(\pi/2) = x_0 + \epsilon$.

(c) By considering the difference between the solutions obtained in parts (a) and (b), comment on the conditioning of this problem.

We first find the solution of the DE. We have $\mathbf{a(t)} = \frac{-1}{\mathbf{t}}$ and $\mathbf{b(t)} = \mathbf{t}\sin(\mathbf{t})$. Thus the integrating factor is $\mathbf{e}^{\int \frac{-1}{t}\mathbf{dt}} = \mathbf{e}^{-\ln(\mathbf{t})} = \frac{1}{\mathbf{t}}$ and then $\int \mathbf{e}^{\int \mathbf{a(t)dt}}\mathbf{b(t)}\ \mathbf{dt} = \int \frac{1}{\mathbf{t}}\mathbf{t}\sin(\mathbf{t})\ \mathbf{dt} = \int \sin(\mathbf{t})\ \mathbf{dt} = -\cos(\mathbf{t})$. Then using Equation (2) we have

$$\mathbf{x(t)} = \frac{-\cos(\mathbf{t}) + \mathbf{C}}{\frac{1}{\mathbf{t}}} = -\mathbf{t}\cos(\mathbf{t}) + \mathbf{Ct}\ .$$

For part (a) we apply the IC: $x(\pi/2) = x_0$ to get ➔ $x_0 = -(\pi/2)\cos(\pi/2) + \mathbf{C}(\pi/2) = \mathbf{C}(\pi/2)$. Now solving for **C** we get $\mathbf{C} = 2x_0/\pi$ and so

$$\mathbf{x(t)} = -\mathbf{t}\cos(\mathbf{t}) + \mathbf{t}\left(\frac{2\mathbf{x_0}}{\pi}\right).$$

For part (b) we apply the IC: $x(\pi/2) = x_0 + \varepsilon$ to get ➔ $x_0 + \varepsilon = -(\pi/2)\cos(\pi/2) + \mathbf{C}(\pi/2) = \mathbf{C}(\pi/2)$. Now solving for **C** we get $\mathbf{C} = 2(x_0 + \varepsilon)/\pi$ and so

$$\mathbf{x(t)} = -\mathbf{t}\cos(\mathbf{t}) + \mathbf{t}\left(\frac{2(\mathbf{x_0} + \varepsilon)}{\pi}\right).$$

For part (c) we compute the difference of the solutions from parts (a) and (b) giving us the following expression

$$\mathbf{t}\left(\frac{2\varepsilon}{\pi}\right).$$

As $\mathbf{t} \to \infty$, even for a small value of **ε** the value of the difference can be very large. Hence a small change in data (here the initial value) causes a large change in the solution **x(t)**. Hence we can say this problem is **ill-conditioned**.