Least Squares additional notes:

Least Squares quadratics: $y = ax^2 + bx + c$

For a data set $S = \{(x_i, y_i) : i=1,2,...,n\}$ we have the over determined system
$ax_i^2 + bx_i + c = y_i$ for I = 1, 2, …,n which in matrix form is

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \Leftrightarrow \mathbf{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}, \text{ where } \mathbf{A} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix}$$

Then to find the least square coefficients of the quadratic we solve linear system

$$\mathbf{A}^{\mathbf{T}}\mathbf{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{A}^{\mathbf{T}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

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Least Squares using power law $\mathbf{y = ax^b}$.

Linearize by taking the logarithm of both sides we get

$$\log(y) = \log(ax^{\mathbf{b}}) = \log(a) + b \log(x)$$

(We could have used the natural logarithm.) For data set $S = \{(x_i, y_i) : i=1,2,...,n\}$ we get the system of equations

$$\log(y_i) = \log(a) + b \log(x_i),\ i\ 1, 2, \ldots, n.$$

What restriction does this place on the data set?

Next let **A = log(a)** and **B = b** and we can write the matrix form of the over determined linear system as

$$\begin{bmatrix} 1 & \mathbf{log}(x_1) \\ 1 & \mathbf{log}(x_2) \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & \mathbf{log}(x_n) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \mathbf{log}(y_1) \\ \mathbf{log}(y_2) \\ \vdots \\ \vdots \\ \mathbf{log}(y_n) \end{bmatrix}$$

We multiply both sides by the transpose of the coefficient matrix and solve the resulting system.

$$\begin{bmatrix} 1 & \mathbf{log}(x_1) \\ 1 & \mathbf{log}(x_2) \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & \mathbf{log}(x_n) \end{bmatrix}^T \begin{bmatrix} 1 & \mathbf{log}(x_1) \\ 1 & \mathbf{log}(x_2) \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & \mathbf{log}(x_n) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{log}(x_1) \\ 1 & \mathbf{log}(x_2) \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & \mathbf{log}(x_n) \end{bmatrix}^T \begin{bmatrix} \mathbf{log}(y_1) \\ \mathbf{log}(y_2) \\ \vdots \\ \vdots \\ \mathbf{log}(y_n) \end{bmatrix}$$

Once we have the values of A and B we can find a and b in the power rule. Since **A = log(a)** and **B = b** it follows that **a = 10^A^** and of course **b = B**. Had we used natural logs, the **a = e^A^**.

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Least Squares using exponential law **$y = ae^{bx}$**.

Linearize by taking the logarithm of both sides we get

$$\ln(y) = \ln(ae^{bx}) = \ln(a) + bx$$

For data set $S = \{(x_i, y_i) : i=1,2,...,n\}$ we get the system of equations

$$\ln(y_i) = \ln(a) + bx_i,\ i\ 1, 2, \ldots, n.$$

What restriction does this place on the data set?

Next let **A = ln(a)** and **B = b** and we can write the matrix form of the over determined linear system as ➔

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \mathbf{log}(y_1) \\ \mathbf{log}(y_2) \\ \vdots \\ \vdots \\ \mathbf{log}(y_n) \end{bmatrix}$$

We multiply both sides by the transpose of the coefficient matrix and solve the resulting system.

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}^T \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}^T \begin{bmatrix} \mathbf{log}(y_1) \\ \mathbf{log}(y_2) \\ \vdots \\ \vdots \\ \mathbf{log}(y_n) \end{bmatrix}$$

Once we have the values of A and B we can find a and b for the exponential law. Since **A = ln(a)** and **B = b** it follows that **a = e^A^** and of course **b = B**.

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Least Squares using the form **$y = \frac{1}{A + Bx}$**.

Linearize by inverting both sides we get

$$\frac{1}{y} = A + Bx$$

For data set $S = \{(x_i, y_i) : i=1,2,...,n\}$ we get the system of equations

$$\frac{1}{y_i} = A + Bx_i\ \ 1, 2, \ldots, n.$$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1/y_1 \\ 1/y_2 \\ \vdots \\ \vdots \\ 1/y_n \end{bmatrix}$$

What restriction does this place on the data set?

In this case we have the over determined linear system as shown. Multiply both sides by the transpose of the coefficient matrix and solve for A and B.