Section 2.4 Newton's Method

Newton's method is the most well known fixed point scheme.

Starting with the root problem **f(x) = 0**, we assume **f** is differentiable and divide both side by **f '(x)**;

$\frac{f(x)}{f'(x)} = 0$. Next multiply both sides by (-1) and add **x** to both sides obtaining

$x = x - \frac{f(x)}{f'(x)}$. Define $g(x) = x - \frac{f(x)}{f'(x)}$ and we have converted the root problem to the fixed point scheme **x = g(x)**. **Warning:** **Newton's method is not GUARANTEED to converge.**

For an initial guess p_0 to a root p of **f**, we define the Newton iteration as

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

Geometrically at the point $(p_n, f(p_n))$ we construct the tangent line to **y = f(x)**,

$$y - f(p_n) = f'(p_n)(x - p_n)$$

and determine the x-intercept as the next approximation p_{n+1}, giving $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$.

Note: each iteration of Newton's method requires TWO separate function evaluations, one for **f** and the other for **f '**. Recall that bisection and false position required only ONE new function evaluation at each step.

Example: Newton's Method in Action

Recall the primary demonstration problem from previous sections: locate the unique zero of the function $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(1, 2)$. To apply Newton's method, the derivative of f is needed. For this problem, $f'(x) = 3x^2 + 4x - 3$. With a starting approximation of $p_0 = 1$, four iterations of Newton's method produce the results

$$\begin{aligned} p_1 &= p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{-1}{4} = 1.25 \\ p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} = 1.2009345794; \\ p_3 &= p_2 - \frac{f(p_2)}{f'(p_2)} = 1.1986958411; \quad \text{and} \\ p_4 &= p_3 - \frac{f(p_3)}{f'(p_3)} = 1.1986912435. \end{aligned}$$

The approximation p_4 is correct to the digits shown and has an absolute error of roughly 1.937×10^{-11}. **Note: 8 function evaluations performed.**

Comparison of methods:

$$f(x) = x^3 + 2x^2 - 3x - 1 \qquad p = -2.9122291785$$

	Bisection	False Position	Newton's Method
Start	(1, 2)	(1,2)	$\mathbf{p_0}$ = 1
Number of Function Evaluations	36	31	8
Absolute Error	≈ 2E-11	≈ 2E-11	≈ 2E-11
Start	(-3, -2)	(-3, -2)	p_0 = -3
Number of Function Evaluations	34	9	6
Absolute Error	4.7E-11	9.1E-11	9.3E-11

Note that Newton's method is **"cheaper"** for the same order of accuracy. Also note that false position was quite accurate using interval (-3, -2).

Newton's method can be quite "sensitive" to the choice of initial guess $\mathbf{p_0}$. Even when Newton's method does converge, it may converge to a value quite far from $\mathbf{p_0}$.

Example: f(x) = sin(x) with **tol** = 1.e-5

$\mathbf{p_0}$	2	1	1.5	1.6	1.58	1.57	1.56
Approximate root	3.14159 ≈ π	0	-12.566 ≈ -4π	31.4159 ≈ 10π	109.955 ≈ 35π	-1253.49 ≈ -399π	-91.1061 ≈ -29π
# of iterations for convergence	6	5	4	8	4	5	4

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Convergence Analysis of Newton's Method

Basically we use the fact that Newton's method is a fixed point scheme. The key is that a good initial guess is needed and then the Newton sequence will converge very rapidly.

Theorem:

Let f be a twice continuously differentiable function on the interval $[a, b]$ with $p \in (a, b)$ and $f(p) = 0$. Further suppose that $f'(p) \neq 0$. Then there exists a $\delta > 0$ such that for any $p_0 \in I = [p - \delta, p + \delta]$, the sequence $\{p_n\}$ generated by Newton's method converges to p.

In the proof we use the fixed point scheme for Newton's method, namely $\mathbf{g}(\mathbf{x}) = \mathbf{x} - \dfrac{\mathbf{f}(\mathbf{x})}{\mathbf{f}'(\mathbf{x})}$.

Step 1. Show g is continuous near p.
Step 2. Show that |g'(x)| is "small" near p.
Step 3. Show that g maps interval I = [p – δ, p + δ] into itself.

Warning and Strategy:

Although this theorem guarantees that δ exists, it may be very small, implying the need for a very good starting approximation to insure convergence of the sequence. For instance, in test problem #2, locating the root of $f(x) = \tan(\pi x) - x - 6$ on the interval $[0, 0.48]$, it can be shown that $\delta \approx 0.02$. It is therefore not uncommon, in practice, to find Newton's method combined with a simple enclosure method. Several iterations of the simple enclosure method are performed to obtain the starting approximation for Newton's method. The interval on which the root has been localized can then be used to test the approximations generated by Newton's method. If one of those approximations is found to fall outside the localizing interval, additional iterations of the simple enclosure method are performed. Newton's method is then restarted with a refined initial estimate of the root. This procedure is repeated as necessary until convergence is obtained.

During the details of proof of the previous theorem it was shown that if $\mathbf{f'(p) \neq 0}$, then for $\mathbf{g(x) = x - \frac{f(x)}{f'(x)}}$, we have **g(p) = p** and $\mathbf{g'(p) = 0}$. Then by the Theorem about the order of convergence of fixed point schemes on p. 91 we determine whether $\mathbf{g''(p)}$ is zero or not. We can show that $\mathbf{g''(p) = \frac{f''(p)}{f'(p)}}$. Since this will not be zero in general Newton's method is quadratically convergent. Since this implies that Newton's method is super linearly convergent our discussion in Section 2.3 implies that an appropriate termination criteria is $\mathbf{|p_n - p_{n-1}| < tol}$.

Example: NEWTON'S METHOD for f = cos(x)-x using derivative f' = -sin(x)-1with initial guess p_0 = 0.5 and tolerance = 1e-015

In 5 iterations we have convergence to: 7.390851332151607e-001

approximations	Absolute error	abs(fix(log10(abs(error))))	
5.000000000000000e-001	2.390851332151607e-001	0	
7.552224171056364e-001	-1.613728389047575e-002	1.000000000000000e+000	**# of accurate**
7.391416661498792e-001	-5.653293471852283e-005	4.000000000000000e+000	**decimal places**
7.390851339208068e-001	-7.056460971099909e-010	9.000000000000000e+000	
7.390851332151607e-001	0	Inf	
7.390851332151607e-001	0	Inf	**implies 15 accurate places**

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What happens to the order of convergence when $\mathbf{f'(p) = 0}$?
(That is, when the multiplicity of the root is > 1.)

If the root **x = p** of **f(x)** has multiplicity $\mathbf{m \geq 2}$, we can show that Newton's method is only linearly convergent for an appropriate initial guess $\mathbf{p_0}$ and the rate of convergence in $O\left(\left(\mathbf{1 - \frac{1}{m}}\right)^{\mathbf{n}}\right)$.

Example: NEWTON'S METHOD for f = (cos(x)-x)^2 using derivative f' = 2*(cos(x)-x)*(-sin(x)-1) with initial guess x0 = 0.5 and tolerance = 1e-015

In 48 iterations we have convergence to: 7.390851332151599e-001

approximations	error	abs(fix(log10(abs(error))))
5.000000000000000e-001	2.390851332151602e-001	0
6.276112085528183e-001	1.114739246623420e-001	0
6.848881310197823e-001	5.419700219537793e-002	1.000000000000000e+000
7.123297687588075e-001	2.675536445635274e-002	1.000000000000000e+000
7.257887251583034e-001	1.329640805685683e-002	1.000000000000000e+000
7.324567215164742e-001	6.628411698685999e-003	2.000000000000000e+000
7.357758118245156e-001	3.309321390644660e-003	2.000000000000000e+000
7.374316858106148e-001	1.653447404545427e-003	2.000000000000000e+000
7.382587118654977e-001	8.264213496624873e-004	3.000000000000000e+000
7.386719980077101e-001	4.131352074501216e-004	3.000000000000000e+000
7.388785844632145e-001	2.065487519457010e-004	3.000000000000000e+000
7.389818635502510e-001	1.032696649092557e-004	3.000000000000000e+000
7.390334995602366e-001	5.163365492366623e-005	4.000000000000000e+000
↓	↓	↓
7.390851332149683e-001	1.919575609576896e-013	1.200000000000000e+001
7.390851332150644e-001	9.581224702515101e-014	1.300000000000000e+001
7.390851332151125e-001	4.773959005888173e-014	1.300000000000000e+001
7.390851332151366e-001	2.364775042451583e-014	1.300000000000000e+001
7.390851332151486e-001	1.165734175856414e-014	1.300000000000000e+001
7.390851332151546e-001	5.662137425588298e-015	1.400000000000000e+001
7.390851332151576e-001	2.664535259100376e-015	1.400000000000000e+001
7.390851332151591e-001	1.110223024625157e-015	1.400000000000000e+001
7.390851332151599e-001	3.330669073875470e-016	1.500000000000000e+001
7.390851332151602e-001	0	Inf

Behavior recognition:

Linear convergence: Once you get one decimal place of accuracy, then every “few” iterations you gain another place of accuracy.

Quadratic convergence: Once you get one decimal place of accuracy, then at each succeeding iteration you almost double the number of accurate decimal places.

Information: Newton’s method can also find complex roots of real functions starting with a complex initial guess.