Section 2.5 Secant Method

Plusses of Newton's Method

- Quadratically convergent for a close enough initial guess.

Minuses of Newton's Method

- Two separate function evaluations per step.
- Requires the derivative.
- Often requires a very good initial guess.

Secant Method:

- Can be considered a variation of Newton's Method or the Method of False Position.
- Does not require the derivative.
- Next approximation is the x-intercept of a line.
- Not an enclosure/bracketing method.
- Needs two initial guesses.
- Needs only one new function evaluation per step.

Description of Secant Method:

Geometrically Newton's method uses the tangent line to **f(x)** at each approximation $\mathbf{p_n}$ to generate the next approximation, hence the derivative value is required. But a tangent line can be approximated by a secant line (a line that connects two points on **f(x)**). With this approach we use

$$f'(p_n) \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}$$

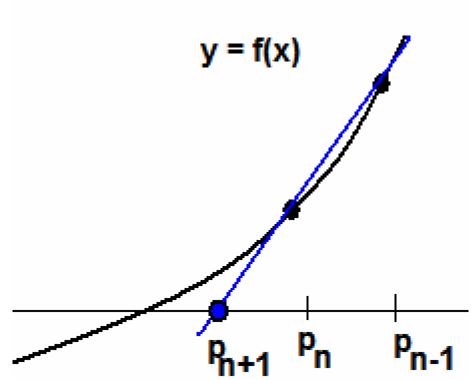
and hence we require two initial guesses in order to start the algorithm. The Secant Method is given by

$$p_{n+1} = p_n - \frac{f(p_n)}{\dfrac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}$$ **where p_0 and p_1 are specified.**

If we save $\mathbf{f(p_n)}$ for the next step the algorithm only requires one new function evaluation per step.

The geometric interpretation of the secant method is as follows:

Given $\mathbf{p_{n-1}}$ and $\mathbf{p_n}$, compute $\mathbf{f}(\mathbf{p_n})$ and $\mathbf{f}(\mathbf{p_{n-1}})$.
Construct the line connecting points $\mathbf{(p_{n-1}, f(p_{n-1}))}$ and $\mathbf{(p_n, f(p_n))}$.
Extend the line to intersect the x-axis.
The x-intercept is taken as $\mathbf{p_{n+1}}$.



<u>Example: The Secant Method in Action</u>

Consider the function $f(x) = x^3 + 2x^2 - 3x - 1$, which we know has a unique zero on the interval $(1, 2)$. Taking $p_0 = 2$ and $p_1 = 1$, the secant method produces

$$p_2 = p_1 - f(p_1)\frac{p_1 - p_0}{f(p_1) - f(p_0)} = 1 - (-1)\frac{1-2}{-1-9} = 1.1.$$

With $p_1 = 1$ and $p_2 = 1.1$, the next secant method approximation is

$$\begin{aligned} p_3 &= p_2 - f(p_2)\frac{p_2 - p_1}{f(p_2) - f(p_1)} \\ &= 1.1 - (-0.549)\frac{1.1 - 1}{-0.549 - (-1)} = 1.2217294900. \end{aligned}$$

The next four iterations produce

$$\begin{aligned} p_4 &= p_3 - f(p_3)\frac{p_3 - p_2}{f(p_3) - f(p_2)} = 1.1964853266; \\ p_5 &= p_4 - f(p_4)\frac{p_4 - p_3}{f(p_4) - f(p_3)} = 1.1986453684; \\ p_6 &= p_5 - f(p_5)\frac{p_5 - p_4}{f(p_5) - f(p_4)} = 1.1986913364; \text{ and} \\ p_7 &= p_6 - f(p_6)\frac{p_6 - p_5}{f(p_6) - f(p_5)} = 1.1986912435. \end{aligned}$$

The approximation p_7 is correct to the digits shown and has an absolute error of roughly 3.907×10^{-12}.

Comparison for this example: For comparable accuracy the Secant Method required more iterations than Newton's method, but 7 function evaluations vs. 8 for Newton's Method.

For good initial guesses the Secant Method can be competitive in "expense" with Newton's Method.

<u>General features of the Secant Method:</u>

- Usually a bit slower than Newton's method. The secant method is super linearly convergent with order of convergence $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.6$.. (Discussed below.)
- It requires fewer function evaluations.
- For some problems the secant method will work when Newton's method doesn't and vice versa.
- The secant method encounters the same type of difficulties as Newton's method.
  - Convergence is not guaranteed when the initial guesses are not close to the root.
  - Method my converge very slowly or not at all.
  - Requires a complex initial guess to approximate a complex root.

Order of Convergence

To determine the order of convergence for the secant method, we need to derive the corresponding error evolution equation. The first step is to subtract the true root, p, from both sides of the recurrence formula for p_{n+1}, yielding

$$p_{n+1} - p = p_n - p - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}.$$

The remaining steps are nearly identical to those used to derive the error evolution equation for the method of false position. The details are therefore left as an exercise. The end result is

$$p_{n+1} - p \approx (p_n - p)(p_{n-1} - p)\frac{f''(p)}{2f'(p) + f''(p)(p_n + p_{n-1} - 2p)}.$$

As p_n and p_{n-1} approach p, the term in the denominator involving the second derivative can be dropped and the leading term in the error is given by

$$|e_{n+1}| \approx C|e_n||e_{n-1}|, \tag{2}$$

where $C = f''(p)/2f'(p)$.

Now, suppose that the Secant method is of order α with asymptotic error constant λ; *i.e.*, successive errors are related by the asymptotic formula $|e_{n+1}| \approx \lambda|e_n|^{\alpha}$. This relationship can also be written as $|e_n| \approx \lambda|e_{n-1}|^{\alpha}$, which, when solved for $|e_{n-1}|$, yields $|e_{n-1}| \approx \lambda^{-1/\alpha}|e_n|^{1/\alpha}$. Substituting for $|e_{n+1}|$ and $|e_{n-1}|$ in (2) leads to

$$\lambda|e_n|^{\alpha} \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha}. \tag{3}$$

Equating powers on $|e_n|$ in (3), it follows that α must satisfy the algebraic equation $\alpha = 1+1/\alpha$. The single positive root of this equation is $\alpha = (1+\sqrt{5})/2$. Hence, the secant method is of order $(1+\sqrt{5})/2 \approx 1.618$. Furthermore, equating the coefficients of $|e_n|$ yields

$$\lambda \approx C^{1/\alpha} = \left(\frac{f''(p)}{2f'(p)}\right)^{\alpha-1}.$$

The analysis we've just completed is based on the assumption that $f'(p) \neq 0$; *i.e.*, p is a simple zero of f. We saw in the previous section that the order of convergence of Newton's method drops to linear when approximating a zero of multiplicity greater than one. In the exercises we will explore whether the same fate befalls the secant method.

If the Secant Method is super linearly convergent, then the stopping criteria $|\mathbf{p_n} - \mathbf{p_{n-1}}| < \mathbf{tol}$ is appropriate based on our discussion in Section 2.3.

<u>Example:</u> f = cos(x)-sin(x) = 0, with tolerance = 1e-008

SECANT METHOD
guesses p_0 = 7 and p_1 = 8
In 20 iterations we have convergence:

7.000000000000000e+000
8.000000000000000e+000
7.078679742108189e+000
7.066940601553275e+000
7.068583493946478e+000
7.068583470577024e+000
7.068583470577035e+000

Note: >> 2*pi+pi/4
ans =
7.0686

NEWTON'S METHOD
with initial guess p_0 = 7

In 3 iterations we have convergence:

7.000000000000000e+000
7.068691205132266e+000
7.068583470576618e+000
7.068583470577035e+000

In this example Secant required 7 function evaluations, while Newton required 6.