

## HANDY CALCULUS THEOREMS<sup>1</sup>

Notation:  $C([a,b])$  = all functions continuous on interval  $[a,b]$

### **Mean Value Theorem**

Let  $f$  be a given function, **continuous** on  $[a,b]$  and **differentiable** on  $(a,b)$ . Then, there exists a point  $\xi \in [a,b]$  such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

From the point of view of numerical methods and analysis, the **Mean Value Theorem (MVT)** is **probably second in importance only to Taylor's Theorem**. Why? Well, consider a slightly reworked form

$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2)$$

Thus, the MVT allows us to replace differences of function values with differences of argument values, if we scale by the derivative of the function.

For example, we can use the MVT to determine that

$$|\cos(x_1) - \cos(x_2)| \leq |x_1 - x_2|$$

EXPLAIN!

Note also that the MVT is simply a special case of Taylor's Theorem, for  $n = 0$ . EXPLAIN!

### **Intermediate Value Theorem**

Let  $f \in C([a,b])$  be given, and assume that  $W$  is a value between  $f(a)$  and  $f(b)$ , i.e., either  $f(a) \leq W \leq f(b)$ , or  $f(b) \leq W \leq f(a)$ . Then there exists a point  $c \in [a,b]$  such that  $f(c) = W$ .

This seems to be a very abstract result; it says that a certain point exists, but doesn't give us much information about its numeric value. (Besides which, we might ask, why do we even care that  $c$  exists?)

This very theorem is the basis for our first method for finding the roots of functions, an algorithm called the **bisection method**.

Moreover, this is the theorem that tells us that a continuous function is one whose graph can be drawn without lifting the pen off the paper, because it says that between any two function values, we have a point on the curve for all possible argument values. **Sometimes very abstract results have very concrete consequences.**

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<sup>1</sup> Handy\_calc\_theorems\_fall\_2009.doc

A related result is the Extreme Value Theorem, which is the basis for the max/min problems that are a staple of most beginning calculus courses.

### Extreme Value Theorem

Let  $f \in C([a,b])$  be given; then there exists a point  $m \in [a,b]$  such that  $f(m) \leq f(x)$  for all  $x \in [a,b]$ , and a point  $M \in [a,b]$  such that  $f(M) \geq f(x)$  for all  $x \in [a,b]$ . Moreover,  $f$  achieves its maximum and minimum values on  $[a,b]$  either at the endpoints  $a$  and  $b$ , or at a critical point.

(Recall that a **critical point** is a point where the first derivative is either undefined or equal to zero. The theorem thus says that either  $M = a$ ,  $M = b$ ,  $f'(M)$  doesn't exist, or  $f'(M) = 0$ , and similarly for  $m$ .)

There are other "mean value theorems," and we need to look at two in particular, as they will come up in our early error analysis.

### Integral Mean Value Theorem

Let  $f$  and  $g$  both be in  $C([a,b])$ , and assume further that  $g$  does not change sign on  $[a,b]$ . Then there exists a point  $\xi \in [a,b]$  such that

$$\int_a^b g(t)f(t)dt = f(\xi) \int_a^b g(t)dt$$

A related result is the Discrete Average Value Theorem:

### Discrete Average Value Theorem

Let  $f \in C([a,b])$  and consider the sum  $S = \sum_{k=1}^n a_k f(x_k)$  where each point

$x_k \in [a, b]$ , and the coefficients satisfy  $a_k \geq 0$ ,  $\sum_{k=1}^n a_k = 1$ . Then there exists a point  $\eta \in [a,b]$

such that  $f(\eta) = S$ , i.e.,  $f(\eta) = \sum_{k=1}^n a_k f(x_k)$ .

Proof : Let  $f_M = \max_{x \in [a,b]} f(x)$  and  $f_m = \min_{x \in [a,b]} f(x)$ . We have that  $f_m \leq f(x) \leq f_M$  so that it follows that  $f_m \leq S \leq f_M$ . EXPLAIN!

Then by the Extreme Value Theorem there is an  $M$  in  $[a,b]$  so that  $f_M = f(M)$  and similarly there is an  $m$  in  $[a,b]$  so that  $f_m = f(m)$ . Hence  $f(m) \leq S \leq f(M)$ . Now apply the Intermediate Value Theorem so that there is an  $\eta$  in  $[a,b]$  so that  $f(\eta) = S$ .

All three of the mean value theorems are useful in that they allow us to simplify certain expressions that will occur in the process of deriving error estimates for our approximate computations.