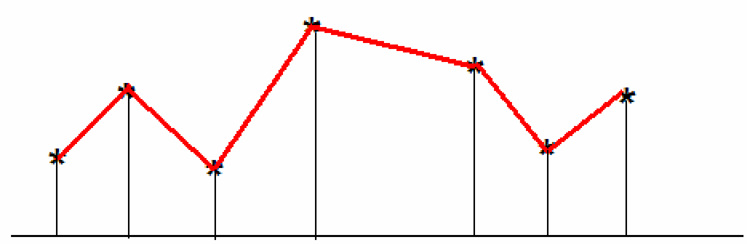
Trapezoidal Rule; An Application of Linear Interpolation[1]

Suppose that function f(x) is only specified by a distinct ordered data set $\{(x_i, f(x_i)) \mid i = 0,1,2,\ldots,n\}$; recall this, means that $x_i < x_{i+1}$. In order to approximate $\int_{x_0}^{x_n} f(x)\,dx$ we construct the piecewise linear interpolant to the data set and compute the sum of the areas under the individual linear interpolants.



Let $p_k(x)$ = linear interpolant on $[x_k, x_{k+1}]$, k = 0, 1, …, n-1, Then

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} p_0(x)\,dx + \int_{x_1}^{x_2} p_1(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} p_{n-1}(x)\,dx$$

CASES:

- **One interval:**

$$\int_{x_0}^{x_1} p(x)\,dx = \int_{x_0}^{x_1} \left(\frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \right) dx = \frac{x_1 - x_0}{2}\left[f(x_0) + f(x_1) \right]$$

This is called the (standard) Trapezoidal Rule.

- **For many intervals of equal length: that is the data set $\{(x_i, f(x_i)) \mid i = 0,1,2,\ldots,n\}$ is such that $x_{i+1} - x_i = h$ for $h > 0$.**

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[1] Trapezoidal_rule.doc \Linz_Wang \ numanal 7/5/2005

(We say the data is equispaced.) The sum of the integrals of the linear interpolants is

$$\frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

This called the **composite Trapezoidal Rule**.

- What do you do if you have lots of points, but they are not equispaced?

Example

Let $f(x) = \sin(x)$ with $x = 0 + i\pi/8$ for I = 0, 1, 2, 3, 4. Then our data set is

$$\{(0,\sin(0)),(\pi/8,\sin(\pi/8)),(\pi/4,\sin(\pi/4)),(3\pi/8,\sin(3\pi/8)),(\pi/2,\sin(\pi/2))\}$$

Then

$$\int_0^{\pi/2} \sin(x)\,dx \approx \frac{\pi/8}{2}\left[\sin(0) + 2\sin(\pi/8) + 2\sin(\pi/4) + 2\sin(3\pi/8) + \sin(\pi/2)\right]$$
$$\approx 0.9871158$$

Of course the true value of the integral 1.

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Error in the Trapezoidal Rule

Basic strategy is to integrate the error in the linear interpolant.

We already know that the error in linear interpolation is

$$f(x) - p(x) = \frac{1}{2}(x - x_0)(x - x_1)f''(\alpha)$$

Integrating this we get

$$\int_{x_0}^{x_1} \left(f(x) - p(x)\right) dx = \int_{x_0}^{x_1} \left(\frac{1}{2}(x - x_0)(x - x_1) f''(\alpha)\right) dx$$

where α is in (x_0, x_1)

Since $(x - x_0)(x - x_1)$ doesn't change sign in $[x_0, x_1]$ we can apply the second mean value theorem for integrals to get

$$= f''(\beta) \int_{x_0}^{x_1} \left(\frac{1}{2}(x - x_0)(x - x_1)\right) dx$$

$$= \frac{1}{2} f''(\beta) \left(\frac{-(x_1 - x_0)^3}{6}\right)$$

Let $h = x_1 - x_0$, then the Error in the Trapezoidal Rule is

$\text{Error(trap)} = -\frac{h^3}{12} f''(\beta)$ for β in (x_0, x_1). The error is $O(h^3)$.

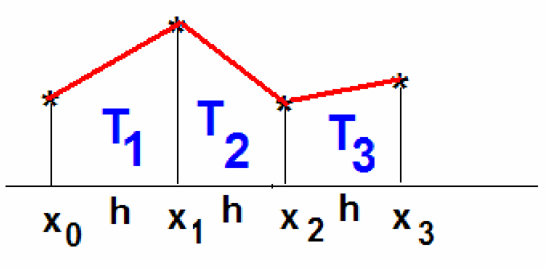
Error in the Composite Trapezoidal Rule

The error in the composite formula is the sum of the errors in each of the standard trapezoidal rule applications over the equispaced intervals. It can be shown that this error can be expressed in the form

$\text{Error(compositetrap)} = -\frac{x_n - x_0}{12} h^2 f''(\beta)$ for β in (x_0, x_n). The error is $O(h^2)$. Note the change in the power of the stepsize h. This is result of accumulated errors over each subinterval.

Example

Rather than prove the error formula for the Composite Trapezoidal Rule in general, we consider the case of three subinterval of equal length.



$$\text{ERROR using Trapezoid \#1} = -\frac{h^3}{12} f''(\beta_1) \quad \text{for } \beta_1 \text{ in } (x_0, x_1)$$

$$\text{ERROR using Trapezoid \#2} = -\frac{h^3}{12} f''(\beta_2) \quad \text{for } \beta_2 \text{ in } (x_1, x_2)$$

$$\text{ERROR using Trapezoid \#3} = -\frac{h^3}{12} f''(\beta_3) \quad \text{for } \beta_3 \text{ in } (x_2, x_3)$$

Adding these errors we

$$-\frac{h^3}{12}\left(f''(\beta_1) + f''(\beta_2) + f''(\beta_3)\right) = -\frac{h^3}{12} 3\left(\frac{1}{3} f''(\beta_1) + \frac{1}{3} f''(\beta_2) + \frac{1}{3} f''(\beta_3)\right)$$

$$= -\frac{h^3}{12} 3f''(\eta)$$

by the discrete average value theorem

$$= -\frac{h^2}{12} 3hf''(\eta) = -\frac{x_3 - x_0}{12} h^2 f''(\eta)$$

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How stable is the composite trapezoidal computation?

Example: **Approximate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$ using the composite trapezoidal rule. Here we will the number of subintervals n used in generating the approximation.**

n	Trap Approximation
1	1
2	1.5
4	1.55
8	1.56558823529412
16	1.56949424724554
32	1.57047080602069
64	1.57071494658749
128	1.57077598174283
256	1.57079124053188
512	1.57079505522914
1024	1.57079600890346

Since the true value of this integral is $\frac{\pi}{2}$ we can compute the absolute error e_n in each approximation. Next we investigate the rate of convergence. Since the trapezoidal rule in $O(h^2)$, we expect that the when n is doubled that the ratios of $\frac{e_n}{e_{2n}}$ should approach 4.

Absolute Error	e_n/e_{2n}
5.707963267948966e-001	NaN
7.079632679489656e-002	8.062513305931052e+000
2.079632679489651e-002	3.404270739401451e+000
5.208091500779055e-003	3.993080150720409e+000
1.302079549351909e-003	3.999825896483289e+000
3.255207742018929e-004	3.999989102214225e+000
8.138020741021990e-005	3.999999318765724e+000
2.034505206904846e-005	3.999999957435648e+000
5.086263020537274e-006	3.999999997424311e+000
1.271565754468185e-006	4.000000002095476e+000
3.178914402823807e-007	3.999999979045242e+000

Using MATLAB routine trap

```
TRAP   Trapezoidal rule to approximate the integral of f(x) over [a,b]
       using n subintervals of equal length.

       Function f is to entered as a string with x as the variable.

   Use in the form  --->  trap(f,a,b,n)  <---

   or in the form   --->  trap   <---
      where the input can be entered step-by-step.
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